Please provide complete and well-written solutions to the following exercises.

Due January 24, in the discussion section.

Homework 2

Exercise 1. Let P,Q be stochastic matrices of the same size. Show that PQ is a stochastic matrix. Conclude that, if r is a positive integer, then P^r is a stochastic matrix.

Exercise 2. Let A,B be events in a sample space. Let $C_1,\ldots,C_n$ be events such that $C_i\cap C_j=\emptyset$ for any $i,j\in\{1,\ldots,n\}$ with $i\neq j$, and such that $\cup_{i=1}^n C_i$ is the whole sample space. Show:

$$\mathbf{P}(A|B)=\sum_{i=1}^n \mathbf{P}(A|B,\,C_i)\mathbf{P}(C_i|B).$$

(Hint: consider using the Total Probability Theorem and that $\mathbf{P}(\cdot|B)$ is a probability law.)

Exercise 3. Let $0<p,q<1$. Let $P=\begin{pmatrix}1-p & p\\ q & 1-q\end{pmatrix}$. Find the (left) eigenvectors of P, and find the eigenvalues of P. By writing any row vector $x\in\mathbf{R}^2$ as a linear combination of eigenvectors of P (whenever possible), find an expression for xP^n for any $n\geq 1$. What is $\lim_{n\to\infty} xP^n$? Is it related to the vector $\pi=(q/(p+q),p/(p+q))$?

Exercise 4. Let $G=(V,E)$ be a graph. Let $|E|$ denote the number of elements in the set E, i.e. $|E|$ is the number of edges of the graph. Prove: $\sum_{x\in V}\deg(x)=2\,|E|$.

Exercise 5. Let A,B be events such that $B\subseteq\{X_0=x_0\}$. Show that $\mathbf{P}(A|B)=\mathbf{P}_{x_0}(A|B)$.

More generally, if A,B are events, show that $\mathbf{P}_{x_0}(A|B)=\mathbf{P}(A|B,X_0=x_0)$.

Exercise 6. Suppose we have a Markov Chain with state space Ω. Let $n\geq 0$, $\ell\geq 1$, let $x_0,\ldots,x_n\in\Omega$ and let $A\subseteq\Omega^\ell$. Using the (usual) Markov property, show that

$$\begin{aligned}&\mathbf{P}((X_{n+1},\ldots,X_{n+\ell})\in A\,|\,(X_0,\ldots,X_n)=(x_0,\ldots,x_n))\\&=\mathbf{P}((X_{n+1},\ldots,X_{n+\ell})\in A\,|\,X_n=x_n).\end{aligned}$$

Then, show that

$$\mathbf{P}((X_{n+1},\ldots,X_{n+\ell})\in A\,|\,X_n=x_n)=\mathbf{P}((X_1,\ldots,X_\ell)\in A\,|\,X_0=x_n).$$

(Hint: it may be helpful to use the Multiplication Rule.)

Exercise 7. Suppose we have a Markov chain $X_0,X_1,\ldots$ with finite state space Ω. Let $y\in\Omega$. Define $L_y:=\max\{n\geq 0\colon X_n=y\}$. Is L_y a stopping time? Prove your assertion.

Exercise 8. Let x,y be points in the state space of a finite Markov Chain $(X_0,X_1,\ldots)$. Let $T_y=\min\{n\geq 1\colon X_n=y\}$ be the first arrival time of y. Let j,k be positive integers. Show that

$$\mathbf{P}_x(T_y>kj\,|\,T_y>(k-1)j)\leq\max_{z\in\Omega}\mathbf{P}_z(T_y>j).$$

(Hint: use Exercise 6)