Math 171, Winter 2017, UCLA — Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ____________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

# Final Exam

This exam contains 15 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 180 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the document.

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 5 | |
| 8 | 15 | |
| 9 | 10 | |
| 10 | 10 | |
| Total: | 100 | |

Do not write in the table to the right. Good luck![a]

**Reference sheet**

Below are some definitions that may be relevant.

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A **(finite or countable) Markov Chain** is a stochastic process $(X_0, X_1, X_2, \ldots)$ together with a finite or countable set $\Omega$, which is called the **state space** of the Markov Chain, and function $P\colon \Omega \times \Omega \to [0,1]$. The random variables $X_0, X_1, \ldots$ take values in the finite set $\Omega$. $P$ is **stochastic**, that is all of its entries are nonnegative and

$$\sum_{y \in \Omega} P(x,y) = 1, \qquad \forall\, y \in \Omega.$$

And the stochastic process satisfies the following **Markov property**: for all $x, y \in \Omega$, for any $n \geq 1$, and for all events $H_{n-1}$ of the form $H_{n-1} = \cap_{k=0}^{n-1}\{X_k = x_k\}$, where $x_k \in \Omega$ for all $0 \leq k \leq n-1$, such that $\mathbf{P}(H_{n-1} \cap \{X_n = x\}) > 0$, we have

$$\mathbf{P}(X_{n+1} = y \,|\, H_{n-1} \cap \{X_n = x\}) = \mathbf{P}(X_{n+1} = y \,|\, X_n = x) = P(x,y).$$

Suppose we have a Markov Chain $X_0, X_1, \ldots$ with state space $\Omega$. Let $y \in \Omega$. Define the **first return time** of $y$ to be the following random variable: $T_y := \min\{n \geq 1 \colon X_n = y\}$. Also, define $\rho_{yy} := \mathbf{P}_y(T_y < \infty)$.

If $\rho_{yy} = 1$, we say the state $y \in \Omega$ is **recurrent**. If $\rho_{yy} < 1$, we say the state $y \in \Omega$ is **transient**. A Markov chain is **irreducible** if any state can reach any other state, with some positive probability, if the chain runs long enough.

We say that $\pi$ is a **stationary distribution** if $\pi(x) \geq 0$ for every $x \in \Omega$, $\sum_{x \in \Omega} \pi(x) = 1$, and if $\pi$ satisfies $\pi = \pi P$ (that is, $\pi(x) = \sum_{y \in \Omega} \pi(y) P(y,x)$ for every $x \in \Omega$.)

Let $P$ be the transition matrix of a finite Markov chain with state space $\Omega$. We say that the Markov chain is **reversible** if there exists a probability distribution $\pi$ on $\Omega$ satisfying the following **detailed balance condition**: $\pi(x)P(x,y) = \pi(y)P(y,x), \qquad \forall\, x, y \in \Omega.$

Let $\mu, \nu$ be probability distributions on a finite state space $\Omega$. We define the **total variation distance** between $\mu$ and $\nu$ to be $||\mu - \nu||_{\mathrm{TV}} := \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|$.

Let $(X_0, X_1, \ldots)$ be a real-valued stochastic process. A **real-valued martingale with respect to** $(X_0, X_1, \ldots)$ is a stochastic process $(M_0, M_1, \ldots)$ such that $\mathbf{E}\,|M_n| < \infty$ for all $n \geq 0$, and for any $m_0, x_0, \ldots, x_n \in \mathbf{R}$,

$$\mathbf{E}(M_{n+1} - M_n | X_n = x_n, \ldots, X_0 = x_0, M_0 = m_0) = 0.$$

A **stopping time** for a martingale $M_0, M_1, \ldots$ is a random variable $T$ taking values in $0, 1, 2, \ldots, \cup\{\infty\}$ such that, for any integer $n \geq 0$, the event $\{T = n\}$ is determined by $M_0, X_0, \ldots, X_n$. More formally, for any integer $n \geq 1$, there is a set $B_n \subseteq \mathbf{R}^{n+2}$ such that

$\{T = n\} = \{(M_0, X_0, \ldots, X_n) \in B_n\}$. Put another way, the indicator function $1_{\{T=n\}}$ is a function of the random variables $M_0, X_0, \ldots, X_n$.

Let $X$ be a random variables on a sample space $\Omega$. Let $A \subseteq \Omega$ with $\mathbf{P}(A) > 0$. Then the **conditional expectation of $X$ given $A$**, denoted $\mathbf{E}(X|A)$ is

$$\mathbf{E}(X|A) := \frac{\mathbf{E}(X \cdot 1_A)}{\mathbf{P}(A)}.$$

Suppose we have a partition of a sample space $\Omega$. That is, we have sets $A_1, \ldots, A_k \subseteq \Omega$ such that $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, and $\cup_{i=1}^k A_i = \Omega$. Denote $\mathcal{A} = \{A_1, \ldots, A_k\}$. Define $\mathbf{E}(X|\mathcal{A})$ to be a random variable that takes the value $\mathbf{E}(X|A_i)$ on the set $A_i$.

Let $\lambda > 0$. Recall that a random variable $T$ is **exponential with parameter** $\lambda$ if $T$ has the density function given by $f_T(x) = \lambda e^{-\lambda x}$ for all $x \geq 0$, and $f_T(x) = 0$ otherwise.

Let $\lambda > 0$. Let $\tau_1, \tau_2, \ldots$ be independent exponential random variables with parameter $\lambda$. Let $T_0 = 0$, and for any $n \geq 1$, let $T_n := \tau_1 + \cdots + \tau_n$. A **Poisson Process** with parameter $\lambda > 0$ is a set of integer-valued random variables $\{N(s)\}_{s \geq 0}$ defined by $N(s) := \max\{n \geq 0 \colon T_n \leq s\}$.

Let $\tau_1, \tau_2, \ldots$ be nonnegative independent identically distributed variables. Let $T_0 = 0$, and for any $n \geq 1$, let $T_n := \tau_1 + \cdots + \tau_n$. A **Renewal process** is a set of integer-valued random variables $\{N(s)\}_{s \geq 0}$ defined by $N(s) := \max\{n \geq 0 \colon T_n \leq s\}$.

Standard Brownian motion with $B(0) = 0$ is uniquely characterized by the following properties:

(i) (Independent increments) For any $0 < t_1 < \cdots < t_n$, the random variables $B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are all independent.

(ii) (Stationary Gaussian increments) for any $0 < s < t$, $B(t) - B(s)$ is a Gaussian random variable with mean zero and variance $t - s$.

(iii) (Continuous Sample Paths) With probability 1, the function $t \mapsto B(t)$ is continuous

1. (10 points) Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1, 2, 3, 4, 5\}$ and with the following transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Classify state 3 as either transient or recurrent.

Is this Markov Chain irreducible? Prove your assertions.

2. (10 points) Consider a non-standard $4 \times 4$ chess board. Let $V$ be a set of vertices corresponding to each square on the board (so $V$ has 16 elements). Any two vertices $x, y \in V$ are connected by an edge if and only if a knight can move from $x$ to $y$. (The knight chess piece moves in an L-shape, so that a single move constitutes two spaces moved along the horizontal axis followed by one move along the vertical axis (or two spaces moved along the vertical axis, followed by one move along the horizontal axis.) Consider the simple random walk on this graph. This Markov chain then represents a knight randomly moving around the chess board. For every space $x$ on the chessboard, compute the expected return time $\mathbf{E}_x T_x$ for that space.

   When you are done, write $\mathbf{E}_x T_x$ for each point $x$ in the chess board below. (You may assume the Markov chain is irreducible.)

| | | | |
|---|---|---|---|
| | | | |
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| | | | |

3. (10 points) Give an example of a random walk on a graph that is not reversible.

4. (10 points) Suppose we have a finite, irreducible, aperiodic Markov chain with transition matrix $P$. Since there exists a unique stationary distribution for this Markov chain, we know that one eigenvalue of $P$ is 1.

   Show that any other eigenvalue $\lambda$ of $P$ satisfies $|\lambda| < 1$.

5. (10 points) Let $X_0 = 0$, and let $a < 0 < b$ be integers. Let $0 < p < 1$ with $p \neq 1/2$. Let $X_1, X_2, \ldots$ be independent identically distributed random variables so that $\mathbf{P}(X_i = 1) = p$ and $\mathbf{P}(X_i = -1) = 1 - p$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n := X_0 + \cdots + X_n$. Define $T := \min\{n \geq 1 \colon Y_n \notin (a, b)\}$.

   Compute $\mathbf{E}T$, in terms of $a, b, p$.

   (Hint: use martingales, somehow. If you use the Optional Stopping Theorem, you do not have to verify that the martingale is bounded.)

   (Second hint: you can freely use the formula $\mathbf{P}(Y_T = a) = \frac{(q/p)^{x_0} - (q/p)^b}{(q/p)^a - (q/p)^b}$, where $q := 1-p$.)

6. (10 points) Let $\{N(s)\}_{s\geq 0}$ be a Poisson Process with parameter $\lambda > 0$. Using the **definition of the Poisson process**, show that, for any $s \geq 0$, $N(s)$ is a Poisson random variable with parameter $\lambda s$.

   (Recall, $X$ is a Poisson random variable with parameter $\lambda > 0$ if $\mathbf{P}(X = n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}$ for all nonnegative integers $n$.)

   (Hint: a gamma distributed random variable $T_n$ with parameters $n$ and $\lambda$ has density

$$f_{T_n}(t) := \begin{cases} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, & \text{if } t \geq 0 \\ 0, & \text{otherwise.)} \end{cases}$$

7. (5 points) Suppose you run a (busy) car wash, and the number of cars that come to the car wash between time 0 and time $s > 0$ is a Poisson process with rate $\lambda = 1$. Suppose every car has either one, two, three, or four people in it. The probability that a car has one, two, three or four people in it is $1/2$, $1/8$, $1/8$ and $1/4$, respectively.

   What is the average number of cars with four people that have arrived by time $s = 60$?

8. (15 points) Let $(X_0, X_1, \ldots)$ be the simple random walk on $\mathbf{Z}$. For any $n \geq 0$, define $M_n = X_n^3 - 3nX_n$. Show that $(M_0, M_1, \ldots)$ is a martingale with respect to $(X_0, X_1, \ldots)$

   Now, fix $m > 0$ and let $T$ be the first time that the walk hits either 0 or $m$. Show that, for any $0 < k \leq m$,
$$\mathbf{E}_k(T \mid X_T = m) = \frac{m^2 - k^2}{3}.$$

   (You can apply the Optional stopping theorem without verifying that the martingale is bounded.)

9. (10 points) Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Let $x_1, \ldots, x_n \in \mathbf{R}$, and let $t_n > \cdots > t_1 > 0$. Show that the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

has a multivariate normal distribution. That is, the joint density of $(B(t_1), \ldots, B(t_n))$ is

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}, \qquad \forall\, x \in \mathbf{R},\ t > 0.$$

10. (10 points) Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Given that $B(1) = 10$, what is the expected length of time after $t = 1$ until $B(t)$ hits either 8 or 12?
- Now, let $\sigma = 2$, and $\mu = -5$. Suppose a commodity has price $X(t) = \sigma B(t) + \mu t$ for any time $t \geq 0$. Given that the price of the commodity is 4 at time $t = 8$, what is the probability that the price is below 1 at time $t = 9$? (You can leave your final answer here as an integral.)

(Scratch paper)

(More scratch paper)