171 Midterm 1 Solutions, Spring 2017[1]

1. Question 1

True/False

(a) Let $\mathbf{P}$ be a probability law on a sample space $\mathcal{C}$. Let $A_1, A_2, \ldots$ be sets in $\mathcal{C}$ which are increasing, so that $A_1 \subseteq A_2 \subseteq \cdots$. Then

$$\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(\cap_{n=1}^{\infty} A_n).$$

FALSE. If $A_1 = \emptyset$, and $A_2 = A_3 = \cdots = \mathcal{C}$, then the left side is 1, while the right side is $\mathbf{P}(\emptyset) = 0$.

(b) The Markov Chain with transition matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has exactly two recurrent states.

FALSE. All three states are recurrent. Since $P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, for any $x \in \{1, 2, 3\}$ we have $\mathbf{P}_x(X_2 = x) = 1$. So, $\mathbf{P}_x(T_x \leq 2) = 1$, and $\mathbf{P}_x(T_x < \infty) = 1$.

(c) Let X, Y be discrete random variables such that

$$\mathbf{P}(X \leq x, Y = y) = \mathbf{P}(X \leq x)\mathbf{P}(Y = y), \qquad \forall\, x, y \in \mathbb{R}.$$

Then

$$\mathbf{P}(X \leq x, Y \leq y) = \mathbf{P}(X \leq x)\mathbf{P}(Y \leq y), \qquad \forall\, x, y \in \mathbb{R}.$$

TRUE. For any $t \in \mathbb{R}$, let $A_t = \{Y = t\}$. Then $A_{t_1} \cap A_{t_2} = \emptyset$ if $t_1 \neq t_2$, and $\cup_{t \leq y} A_t = \{Y \leq y\}$, so

$$\sum_{t \leq y} \mathbf{P}(X \leq x, Y = t) = \sum_{t \leq y} \mathbf{P}(\{X \leq x\} \cap A_t) = \mathbf{P}(\{X \leq x\} \cap (\cup_{t \leq y} A_t)) = \mathbf{P}(X \leq x, Y \leq y).$$

Similarly, $\sum_{t \leq y} \mathbf{P}(Y = t) = \mathbf{P}(Y \leq y)$. So, summing both sides of the equality $\mathbf{P}(X \leq x, Y = t) = \mathbf{P}(X \leq x)\mathbf{P}(Y = t)$ over all $t \leq y$ proves the assertion.

2. Question 2

For any $x \in \mathbb{R}$, define

$$\phi(x) := \max\Big(-x - 1,\, 0,\, x - 1\Big).$$

Prove that $\phi \colon \mathbb{R} \to \mathbb{R}$ is convex.

(In this problem, unlike the other problems, you are allowed to use results from the homework.)

Solution. From Homework 1, Exercise 1, it suffices to show: for any $y \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $L(x) := a(x - y) + \phi(y)$ satisfies $L(y) = \phi(y)$ and $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$. We break into three cases.

Case 1. $y \in [-1, 1]$. In this case we choose $a = 0$. Then $L(x) = 0$ for all $x \in \mathbb{R}$, $L(y) = \phi(y) = 0$, and $L(x) = 0 \leq \phi(x)$ for all $x \in \mathbb{R}$ by definition of ϕ.

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Case 2. $y > 1$. In this case we choose $a = 1$. Then $L(x) = (x-y)+(y-1) = x-1$ for all $x \in \mathbb{R}$, $L(y) = y-1 = \phi(y)$, and $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$, since $L(x) = \phi(x)$ when $x \geq 1$, and if $x < 1$, then $L(x) < 0 \leq \phi(x)$ since $\phi(x) \geq 0$ by definition of ϕ.

Case 3. $y < -1$. In this case we choose $a = -1$. Then $L(x) = -(x-y)+(-y-1) = -x-1$ for all $x \in \mathbb{R}$, $L(y) = -y-1 = \phi(y)$, and $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$, since $L(x) = \phi(x)$ when $x \leq -1$, and if $x > -1$, then $L(x) < 0 \leq \phi(x)$ since $\phi \geq 0$ by definition of ϕ.

3. Question 3

Suppose we have a Markov chain $X_0, X_1, \ldots$ with finite state space Ω. Let $y \in \Omega$. Define $L_y := \max\{n \geq 0 \colon X_n = y\}$. Is L_y a stopping time? Prove your assertion.

Solution. No, L_y is not a stopping time. We argue by contradiction. Let $\Omega := \{1, 2\}$. If L_1 were a stopping time, then there exists $B \subseteq \Omega^2$ such that $\{L_1 = 1\} = \{(X_0, X_1) \in B\}$. But $\{L_1 = 1\} = \{X_1 = 1,\, 2 = X_2 = X_3 = X_4 = \cdots\}$. That is, the B as defined before does not exist.

4. Question 4

Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1, 2, 3, 4, 5\}$ and with the following transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Classify state 3 as either transient or recurrent.

Is this Markov Chain irreducible? Prove your assertions.

Solution. State 3 is transient since $P(3, 2) > 0$, while $P^n(2, 3) = 0$ for all $n \geq 1$. In fact, it follows by induction on n and the definition of P that $P^n(i, j) = 0$ for all $1 \leq i \leq 2$ and $3 \leq j \leq 5$. By the definition of matrix multiplication, by the inductive hypothesis, and by definition of P, we have

$$P^{n+1}(i, j) = \sum_{k=1}^{5} P^n(i, k)P(k, j) = \sum_{k=1}^{2} P^n(i, k)P(k, j) = \sum_{k=1}^{2} P^n(i, k) \cdot 0 = 0.$$

So, if $X_1 = 2$, we know that $X_n \in \{1, 2\}$ for all $n \geq 1$ with probability 1. Therefore,

$$\mathbf{P}_3(T_3 = \infty) \geq \mathbf{P}_3(X_1 = 2, X_n \in \{1, 2\}\, \forall\, n \geq 2) = \mathbf{P}_3(X_1 = 2) = P(3, 2) = 1/3 > 0.$$

The Markov chain is not irreducible, since as we mentioned above, $P^n(1, 3) = 0$ for all $n \geq 1$.

5. Question 5

Give an example of a Markov chain on the state space $\Omega = \{1, 2\}$ such that state 1 is recurrent and state 2 is transient. Prove your assertions.

Solution. Define

$$P := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then P is a stochastic matrix, so P defines a Markov chain on $\{1, 2\}$. Since $P(1, 1) = 1$, $\mathbf{P}_1(T_1 < \infty) \geq \mathbf{P}_1(T_1 = 1) = P(1, 1) = 1$. That is, $\mathbf{P}_1(T_1 < \infty) = 1$, so the state 1 is recurrent. On the other hand, $\mathbf{P}_2(T_2 = \infty) = \mathbf{P}_2(1 = X_1 = X_2 = X_3 = \cdots) = \lim_{n\to\infty} P(2, 1)[P(1, 1)]^n = 1$. That is, $\mathbf{P}_2(T_2 < \infty) = 0$, so that state 2 is transient.