Math 171, Winter 2017, UCLA Instructor: Steven Heilman

Name: ____________________ UCLA ID: ______________ Date: ____________

Signature: ____________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 1

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	9	
2	10	
3	10	
4	10	
5	10	
Total:	49	

Do not write in the table to the right. Good luck![a]

Reference sheet

Below are some definitions that may be relevant.

A **finite Markov Chain** is a stochastic process $(X_0, X_1, X_2, \ldots)$ together with a finite set Ω, which is called the **state space** of the Markov Chain, and an $|\Omega| \times |\Omega|$ real matrix P. The random variables $X_0, X_1, \ldots$ take values in the finite set Ω. The matrix P is **stochastic**, that is all of its entries are nonnegative and

$$\sum_{y \in \Omega} P(x, y) = 1, \qquad \forall\, x \in \Omega.$$

And the stochastic process satisfies the following **Markov property**: for all $x, y \in \Omega$, for any $n \geq 1$, and for all events H_{n-1} of the form $H_{n-1} = \cap_{k=0}^{n-1} \{X_k = x_k\}$, where $x_k \in \Omega$ for all $0 \leq k \leq n-1$, such that $\mathbf{P}(H_{n-1} \cap \{X_n = x\}) > 0$, we have

$$\mathbf{P}(X_{n+1} = y \,|\, H_{n-1} \cap \{X_n = x\}) = \mathbf{P}(X_{n+1} = y \,|\, X_n = x) = P(x, y).$$

Suppose we have a Markov Chain $X_0, X_1, \ldots$ with state space Ω. Let $y \in \Omega$. Define the **first return time** of y to be the following random variable: $T_y := \min\{n \geq 1 \colon X_n = y\}$. Also, define $\rho_{yy} := \mathbf{P}_y(T_y < \infty)$.

If $\rho_{yy} = 1$, we say the state $y \in \Omega$ is **recurrent**. If $\rho_{yy} < 1$, we say the state $y \in \Omega$ is **transient**.

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (3 points) Let $\mathbf{P}$ be a probability law on a sample space $\mathcal{C}$. Let $A_1, A_2, \ldots$ be sets in $\mathcal{C}$ which are increasing, so that $A_1 \subseteq A_2 \subseteq \cdots$. Then

   $$\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(\cap_{n=1}^{\infty} A_n).$$

   TRUE FALSE (circle one)

   (b) (3 points) The Markov Chain with transition matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has exactly two recurrent states.

   TRUE FALSE (circle one)

   (c) (3 points) Let X, Y be discrete random variables such that

   $$\mathbf{P}(X \leq x, Y = y) = \mathbf{P}(X \leq x)\mathbf{P}(Y = y), \qquad \forall\, x, y \in \mathbf{R}.$$

   Then

   $$\mathbf{P}(X \leq x, Y \leq y) = \mathbf{P}(X \leq x)\mathbf{P}(Y \leq y), \qquad \forall\, x, y \in \mathbf{R}.$$

   TRUE FALSE (circle one)

2. (10 points) For any $x \in \mathbf{R}$, define

$$\phi(x) := \max\Big(-x-1,\, 0,\, x-1\Big).$$

Prove that $\phi\colon \mathbf{R} \to \mathbf{R}$ is convex.

(In this problem, unlike the other problems, you are allowed to use results from the homework.)

3. (10 points) Suppose we have a Markov chain $X_0, X_1, \dots$ with finite state space Ω. Let $y \in \Omega$. Define $L_y := \max\{n \geq 0\colon X_n = y\}$. Is L_y a stopping time? Prove your assertion.

4. (10 points) Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1, 2, 3, 4, 5\}$ and with the following transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Classify state 3 as either transient or recurrent.

Is this Markov Chain irreducible? Prove your assertions.

5. (10 points) Give an example of a Markov chain on the state space $\Omega = \{1, 2\}$ such that state 1 is recurrent and state 2 is transient. Prove your assertions.

(Scratch paper)