Please provide complete and well-written solutions to the following exercises.

Due December 1, in the discussion section.

Homework 8

Exercise 1 (Scaling Invariance)**.** Let $a > 0$. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. For any $t > 0$, define $X(t) := \frac{1}{\sqrt{a}}B(at)$. Then $\{X(t)\}_{t\geq 0}$ is also a standard Brownian motion.

Exercise 2. Let $x_1, \ldots, x_n \in \mathbf{R}$, and if $t_n > \cdots > t_1 > 0$. Using the independent increment property, show that the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

has a multivariate normal distribution. That is, the joint density of $(B(t_1), \ldots, B(t_n))$ is

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1)f_{t_2-t_1}(x_2 - x_1)\cdots f_{t_n-t_{n-1}}(x_n - x_{n-1})$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/(2t)}, \qquad \forall x \in \mathbf{R},\, t > 0.$$

Exercise 3. Let X be a Gaussian random variable with mean 0 and variance $\sigma_X^2 > 0$. Let Y be a Gaussian random variable with mean 0 and variance $\sigma_Y^2 > 0$. Assume that X and Y are independent. Show that $X + Y$ is also a Gaussian random variable with mean 0 and variance $\sigma_X^2 + \sigma_Y^2$.

(Hint: write an expression for $\mathbf{P}(X + Y \leq t)$, $t \in \mathbf{R}$, then take a derivative in t.)

Exercise 4. Let $A := \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$. Find $\inf(A)$. Note that $\inf(A)$ exists, but A has no minimum element. The infimum is better to work with for this reason.

Exercise 5. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Then $\{(B(t))^2 - t\}_{t\geq 0}$ is a (continuous-time) martingale in the following sense: it $t > s > 0$, and if $s > s_n > \cdots > s_1 > 0$, and $x_1, \ldots, x_n \in \mathbf{R}$, then

$$\mathbf{E}((B(t))^2 - t - ((B(s))^2 - s) \,|\, B(s_n) = x_n, \ldots, B(s_1) = x_1) = 0.$$

Exercise 6. Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Given that $B(1) = 10$, what is the expected length of time after $t = 1$ until $B(t)$ hits either 8 or 12?
- Now, let $\sigma = 2$, and $\mu = -5$. Suppose a commodity has price $X(t) = \sigma B(t) + \mu t$ for any time $t \geq 0$. Given that the price of the commodity is 4 at time $t = 8$, what is the probability that the price is below 1 at time $t = 9$?
- Suppose a stock has a price $S(t) = 4e^{B(t)}$ for any $t \geq 0$. That is, the stock moves according to Geometric Brownian Motion. What is the probability that the stock reaches a price of 7 before it reaches a price of 2?

Exercise 7. Fix $x > 0$

- Show the bound $\mathbf{P}(-x < B(t) < x) \geq \frac{x}{20\sqrt{t}}$ holds for all $t > x^2$.
- Show that $\mathbf{E}T_x = \infty$. (Recall we observed something similar for the simple random walk on $\mathbf{Z}$.)

Exercise 8. Let $\{X(s)\}_{s\geq 0}$ be a standard Brownian motion with drift μ and variance σ^2. For any $t > s > 0$, show that $X(t) - X(s)$ is a Gaussian random variable with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$.

Exercise 9. Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and drift $\mu \in \mathbf{R}$. Fix $\lambda \in \mathbf{R}$. Then $\{Y(t)\}_{t\geq 0} = \{e^{\lambda X(t) - (\lambda\mu + \lambda^2\sigma^2/2)t}\}_{t\geq 0}$ is a (continuous-time) martingale in the following sense: it $t > s > 0$, and if $s > s_n > \cdots > s_1 > 0$, and $x_1, \ldots, x_n \in \mathbf{R}$, then

$$\mathbf{E}(Y(t) - Y(s) \mid B(s_n) = x_n, \ldots, B(s_1) = x_1) = 0.$$

Exercise 10. Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and negative drift $\mu < 0$. Let $a < 0 < b$. Let $T := \inf\{t \geq 0 \colon X(t) \in \{a, b\}\}$. Let $\alpha := 2\,|\mu|\,/\sigma^2$. Show that

$$\mathbf{E}T = \frac{1}{\mu} \cdot \frac{b(1 - e^{\alpha a}) + a(e^{\alpha b} - 1)}{e^{\alpha b} - e^{\alpha a}}$$

Exercise 11. Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and negative drift $\mu < 0$. Let $a < 0$. Let $T_a := \inf\{t \geq 0 \colon X(t) = a\}$. Let $\alpha := 2\,|\mu|\,/\sigma^2$. Show that

$$\mathbf{E}T_a = \frac{a}{\mu}.$$

Exercise 12 (Optional)**.** Write a computer program to simulate standard Brownian motion. More specifically, the program should simulate a random walk on $\mathbf{Z}$ with some small step size such as .002. (That is, simulate $B_k(t)$ when $k = 500^2$ and, say, $0 \leq t \leq 1$.)

Exercise 13 (Optional)**.** The following exercise assumes familiarity with Matlab and is derived from Cleve Moler's book, Numerical Computing with Matlab.

The file `brownian.m` plots the evolution of a cloud of particles that starts at the origin and diffuses in a two-dimensional random walk, modeling the Brownian motion of gas molecules.

(a) Modify `brownian.m` to keep track of both the average and the maximum particle distance from the origin. Using loglog axes, plot both sets of distances as functions of n, the number of steps. You should observe that, on the log-log scale, both plots are nearly linear. Fit both sets of distances with functions of the form $cn^{1/2}$. Plot the observed distances and the fits, using linear axes.

(b) Modify `brownian.m` to model a random walk in three dimensions. Do the distances behave like $n^{1/2}$?

The program `brownian.m` appears below.

```
% BROWNIAN   Two-dimensional random walk.
```

```
%   What is the expansion rate of the cloud of particles?

shg
clf
set(gcf,'doublebuffer','on')
delta = .002;
x = zeros(100,2);
h = plot(x(:,1),x(:,2),'.');
axis([-1 1 -1 1])
axis square
stop = uicontrol('style','toggle','string','stop');
while get(stop,'value') == 0
   x = x + delta*randn(size(x));
   set(h,'xdata',x(:,1),'ydata',x(:,2))
   drawnow
end
set(stop,'string','close','value',0,'callback','close(gcf)')
```