Please provide complete and well-written solutions to the following exercises.

Due November 17, in the discussion section.

Homework 6

Exercise 1. Let $\lambda > 0$. Let $\tau_1, \tau_2, \ldots$ be independent exponential random variables with parameter λ. For any $n \geq 1$, let $T_n = \tau_1 + \cdots + \tau_n$. Fix positive integers $n_k > \cdots > n_1$ and positive real numbers $t_k > \cdots > t_1$. Then

$$f_{T_{n_k},\ldots,T_{n_1}}(t_k, \ldots, t_1) = f_{T_{(n_k - n_{k-1})}}(t_k - t_{k-1}) \cdots f_{T_{(n_2 - n_1)}}(t_2 - t_1) f_{T_{n_1}}(t_1).$$

(Hint: just try to case $k = 2$ first, and use a conditional density function.)

Exercise 2. Let $s, t > 0$ and let m, n be nonnegative integers. Let $0 < t_m < t_{m+1} < t_{m+n} < t_{m+n+1}$, and define (using the notation of Exercise 1),

$$g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) := f_{T_1}(t_{m+n+1} - t_{m+n}) f_{T_{n-1}}(t_{m+n} - t_{m+1}) f_{T_1}(t_{m+1} - t_m) f_{T_m}(t_m).$$

Let $\{N(s)\}_{s \geq 0}$ be a Poisson Process with parameter $\lambda > 0$. Show that

$$\begin{aligned}&\mathbf{P}(N(s+t) = m+n,\, N(s) = m)\\ &= \int_0^s \left(\int_s^{s+t} \left(\int_{t_{m+1}}^{s+t} \left(\int_{s+t}^{\infty} g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) dt_{m+n+1} \right) dt_{m+n} \right) dt_{m+1} \right) dt_m.\end{aligned}$$

(Hint: use the joint density, and then use Exercise 1.)

Exercise 3. Let $Y_1, Y_2, \ldots$ be independent identically distributed random variables. Let N be an independent, nonnegative integer-valued random variable. Let $S = Y_1 + \cdots + Y_N$, where $S := 0$ if $N = 0$.

- If $\mathbf{E}\,|Y_1| < \infty$ and $\mathbf{E}N < \infty$, then $\mathbf{E}S = (\mathbf{E}N)(\mathbf{E}Y_1)$.
- If $\mathbf{E}Y_1^2 < \infty$ and $\mathbf{E}N^2 < \infty$, then $\mathrm{var}(S) = (\mathbf{E}N)(\mathrm{var}(Y_1)) + (\mathbf{E}Y_1)^2(\mathrm{var}(N))$.
- If N is a Poisson random variable with parameter $\lambda > 0$, then $\mathrm{var}(S) = \lambda \mathbf{E}Y_1^2$.

(Hint: for the second part, use $\mathbf{E}(S^2 | N = n) = n \cdot \mathrm{var}(Y_1) + (n\mathbf{E}Y_1)^2$. Use this to compute $\mathbf{E}S^2$. Then compute $\mathrm{var}(S)$.)

Exercise 4. Suppose the number of students going to a restaurant in Ackerman in a single day has a Poisson distribution with mean 500. Suppose each student spends an average of \$10 with a standard deviation of \$5. What is the average revenue of the restaurant in one day? What is the standard deviation of the revenue in one day? (The amounts spent by the students are independent identically distributed random variables.)