Please provide complete and well-written solutions to the following exercises.

Due November 3, in the discussion section.

Homework 4

Exercise 1. Let Ω be a finite state space. This exercise demonstrates that the total variation distance is a metric. That is, the following three properties are satisfied:

- $||\mu-\nu||_{\mathrm{TV}} \geq 0$ for all probability distributions μ,ν on Ω, and $||\mu-\nu||_{\mathrm{TV}}=0$ if and only if $\mu=\nu$.
- $||\mu-\nu||_{\mathrm{TV}}=||\nu-\mu||_{\mathrm{TV}}$
- $||\mu-\nu||_{\mathrm{TV}} \leq ||\mu-\eta||_{\mathrm{TV}}+||\eta-\nu||_{\mathrm{TV}}$ for all probability distributions μ,ν,η on Ω.

(Hint: you may want to use the triangle inequality for real numbers: $|x-y| \leq |x-z|+|z-y|$, $\forall\, x,y,z \in \mathbf{R}$.)

Exercise 2. Let μ,ν be probability distributions on a finite state space Ω. Then

$$||\mu-\nu||_{\mathrm{TV}}=\frac{1}{2}\sum_{x\in\Omega}|\mu(x)-\nu(x)|\,.$$

(Hint: consider the set $A=\{x\in\Omega\colon \mu(x)\geq\nu(x)\}$.)

Exercise 3. Let $(X_0,X_1,\ldots)$ be the simple random walk on $\mathbb{Z}$. Show that $\mathbf{P}_0(X_n=0)$ decays like $1/\sqrt{n}$ as $n\to\infty$. That is, show

$$\lim_{n\to\infty}\sqrt{2n}\,\mathbf{P}_0(X_{2n}=0)=\sqrt{\frac{2}{\pi}}.$$

Also, show the upper bound

$$\mathbf{P}_0(X_n=k)\leq\frac{10}{\sqrt{n}},\qquad \forall\, n\geq 0,\, k\in\mathbb{Z}.$$

(Hint 1: first consider the case $n=2r$ for $r\in\mathbb{Z}$. It may be helpful to show that $\binom{2r}{r+j}$ is maximized when $j=0$. To eventually deal with k odd, just condition on the first step of the walk.)

(Hint 2: you can freely use **Stirling's formula**:

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}(n/e)^n}=1.$$

Or, there is a more precise estimate: for any $n\geq 3$, there exists $1/(12n+1)\leq\varepsilon_n\leq 1/(12n)$ such that

$$n!=\sqrt{2\pi}e^{-n}n^{n+1/2}e^{\varepsilon_n}.)$$

Exercise 4. Show that every state in the simple random walk on $\mathbb{Z}$ is recurrent. (You should show this statement for any starting location of the Markov chain.)

Then, find a nearest-neighbor random walk on $\mathbb{Z}$ such that every state is transient.

Exercise 5. For the simple random walk on $\mathbb{Z}$, show that $\mathbf{E}_0 T_0 = \infty$. Conclude that, for any $x, y \in \mathbb{Z}$, $\mathbf{E}_x T_y = \infty$.

Exercise 6. Let $(X_0, X_1, \ldots)$ be the "corner walk" on $\mathbb{Z}^2$. The transitions are described as follows. From any point $(x, y) \in \mathbb{Z}^2$, the Markov chain adds any of the following four vector to (x, y) each with probability 1/4: $\{(1,1,), (1,-1), (-1,1), (-1,-1)\}$. Using that the coordinates of this walk are each independent simple random walks on $\mathbb{Z}$, conclude that there exists $c > 0$ such that

$$\lim_{n\to\infty} n\mathbf{P}_{(0,0)}(X_{2n} = (0,0)) = c.$$

That is, $\mathbf{P}_{(0,0)}(X_{2n} = (0,0))$ is about c/n, when n is large.

Now, note that the usual nearest-neighbor simple random walk on $\mathbb{Z}^2$ is a rotation of the corner walk by an angle of $\pi/4$. So, the above limiting statement also holds for the simple random walk on $\mathbb{Z}^2$.

Exercise 7. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X\colon [0,1] \to \mathbf{R}$ be a random variable such that $X(t) = t^2$ for all $t \in [0,1]$. Let

$$\mathcal{A} = \{[0,1/4), [1/4,1/2), [1/2,3/4), [3/4,1]\}$$

Compute explicitly the function $\mathbf{E}(X|\mathcal{A})$. (It should be constant on each of the partition elements.) Draw the function $\mathbf{E}(X|\mathcal{A})$ and compare it to a drawing of X itself.

Now, for every integer $k > 1$, let $s = 2^{-k}$, and let $\mathcal{A}_k := \{[0,s), [s,2s), [2s,3s), \ldots, [1-2s, 1-s), [1-s, 1)\}$. Try to draw $\mathbf{E}(X|\mathcal{A}_k)$. Convince yourself of the following fact (you can prove it if you want, but you do not have to): for every $t \in [0,1]$

$$\lim_{k\to\infty} \mathbf{E}(X|\mathcal{A}_k)(t) = X(t).$$

The purpose of this exercise is to demonstrate that $\mathbf{E}(X|\mathcal{A})$ is given by averaging X over each partition element, such that $\mathbf{E}(X|\mathcal{A})$ is constant on each partition element of $\mathcal{A}$.

Exercise 8. Let X be a random variable with finite variance, and let $t \in \mathbf{R}$. Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(t) = \mathbf{E}(X-t)^2$. Show that the function f is uniquely minimized when $t = \mathbf{E}X$. That is, $f(\mathbf{E}X) < f(t)$ for all $t \in \mathbf{R}$ such that $t \neq \mathbf{E}X$. Put another way, setting t to be the mean of X minimizes the quantity $\mathbf{E}(X-t)^2$ uniquely.

The conditional expectation, being a piecewise version of taking an average, has a similar property. Let $A_1, \ldots, A_k \subseteq \Omega$ such that $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, and $\cup_{i=1}^k A_i = \Omega$. Write $\mathcal{A} = \{A_1, \ldots, A_k\}$. By definition, for each $1 \leq i \leq k$, $\mathbf{E}(X|\mathcal{A})$ is constant on A_i. Now, let Y be any other random variable such that, for each $1 \leq i \leq k$, Y is constant on A_i. Show that the quantity $\mathbf{E}(X-Y)^2$ is uniquely minimized by such a Y only when $Y = \mathbf{E}(X|\mathcal{A})$.

Exercise 9. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X\colon [0,1] \to \mathbf{R}$ be a random variable such that $X(t) = t^2$ for all $t \in [0,1]$. For every integer $k > 1$, let $s = 2^{-k}$, let $\mathcal{A}_k := \{[0,s), [s,2s), [2s,3s), \ldots, [1-2s, 1-s), [1-s, 1)\}$, and let $M_k := \mathbf{E}(X|\mathcal{A}_k)$. Show that the increments $M_2 - M_1, M_3 - M_2, \ldots$ are orthogonal in the following sense. For any $i, j \geq 1$ with $i \neq j$,

$$\mathbf{E}(M_{i+1} - M_i)(M_{j+1} - M_j) = 0.$$

This property is sometimes called **orthogonality of martingale increments**. This property holds for many martingales, but we will not prove this.