Please provide complete and well-written solutions to the following exercises.

Due October 27, in the discussion section.

Homework 3

Exercise 1 (Knight Moves)**.** Consider a standard 8×8 chess board. Let V be a set of vertices corresponding to each square on the board (so V has 64 elements). Any two vertices $x, y \in V$ are connected by an edge if and only if a knight can move from x to y. (The knight chess piece moves in an L-shape, so that a single move constitutes two spaces moved along the horizontal axis followed by one move along the vertical axis (or two spaces moved along the vertical axis, followed by one move along the horizontal axis.) Consider the simple random walk on this graph. This Markov chain then represents a knight randomly moving around a chess board. For every space x on the chessboard, compute the expected return time $\mathbf{E}_x T_x$ for that space. (It might be convenient to just draw the expected values on the chessboard itself.)

Exercise 2 (Simplified Monopoly)**.** Let $\Omega = \{1, 2, \ldots, 10\}$. We consider Ω to be the ten spaces of a circular game board. You move from one space to the next by rolling a fair six-sided die. So, for example $P(1, k) = 1/6$ for every $2 \leq k \leq 7$. More generally, for every $j \in \Omega$ with $j \neq 5$, $P(j, k) = 1/6$ if $k = (j + i) \bmod 10$ for some $1 \leq i \leq 6$. Finally, the space 5 forces you to return to 1, so that $P(5, 1) = 1$. (Note that $\bmod 10$ denotes arithmetic modulo 10, so e.g. $7 + 5 = 2 \bmod 10$.)

Using a computer, find the unique stationary distribution of this Markov chain. Which point has the highest stationary probability? The lowest?

Compare this stationary distribution to the stationary distribution that arises from the doubly stochastic matrix: for all $j \in \Omega$, $P(j, k) = 1/6$ if $k = (j + i) \bmod 10$ for some $1 \leq i \leq 6$. (See Exercise 5.)

Exercise 3. Give an example of a Markov chain where there are at least two different stationary distributions.

Exercise 4. Is there a finite Markov chain where no stationary distribution exists? Either find one, or prove that no such finite Markov chain exists.

(If you want to show that no such finite Markov chain exists, you are allowed to just prove the weaker assertion that: for every stochastic matrix P, there always exists a nonzero vector π with $\pi = \pi P$.)

Exercise 5. Let P be the transition matrix for a finite Markov chain with state space Ω. We say that the matrix P is **doubly stochastic** if the columns of P each sum to 1. (Since P is a transition matrix, each of its rows already sum to 1.) Let π such that $\pi(x) = 1/|\Omega|$ for all $x \in \Omega$. That is, π is uniform on Ω. Show that $\pi = \pi P$.

Exercise 6. Give an example of a random walk on a graph that is not reversible.

Exercise 7. Let P be the transition matrix of a finite, irreducible, reversible Markov chain with state space Ω and stationary distribution π. Let $f, g \in \mathbf{R}^{|\Omega|}$ be column vectors. Consider the following bilinear function on f, g, which is referred to as an inner product (or dot product):

$$\langle f, g \rangle := \sum_{x \in \Omega} f(x) g(x) \pi(x).$$

Show that P is self-adjoint (i.e. symmetric) in the sense that

$$\langle f, Pg \rangle = \langle Pf, g \rangle.$$

In particular (for those that have taken 115A), the spectral theorem implies that all eigenvalues of P are real.

Finally, find a transition matrix P such that at least one eigenvalue of P is not real.

Exercise 8 (Ehrenfest Urn Model)**.** Suppose we have two urns and n spheres. Each sphere is in either of the first or the second urn. At each step of the Markov chain, one of the spheres is chosen uniformly and random and moved from its current urn to the other urn. Let X_n be the number of spheres in the first urn at time n. Then the transition matrix defining the Markov chain is

$$P(j,k) = \begin{cases} \frac{n-j}{n} & \text{, if } k = j+1 \\ \frac{j}{n} & \text{, if } k = j-1 \\ 0 & \text{, otherwise.} \end{cases}$$

Show that the unique stationary distribution for this Markov chain is a binomial with parameters n and $1/2$.

Exercise 9. Let $V = \{0,1\}^n$ be a set of vertices. We construct a graph from V as follows. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \{0,1\}^n$. Then x and y are connected by an edge in the graph if and only if $\sum_{i=1}^n |x_i - y_i| = 1$. That is, x and y are connected if and only if they differ by a single coordinate.

For any $x \in V$, define $f(x) = \sum_{i=1}^n x_i$, $f \colon V \to \{0, 1, \ldots, n\}$. Given $x \in V$, we identify x with the state in the Ehrenfest urn model where the first urn has exactly $f(x)$ spheres. Show that the Ehrenfest urn model is a **projection** of the simple random walk on V in the following sense. The probability that $x \in V$ transitions to any state $z \in V$ such that $y = f(z)$ is equal to: the probability that Ehrenfest model with state $f(x)$ transitions to state y.

Moreover, the unique stationary distribution for the simple random walk on V can be projected to give the unique stationary distribution in the Ehrenfest model. That is, if π is the unique stationary distribution for the simple random walk on V, and if for any $A \subseteq \{0, 1, \ldots, n\}$, we define $\mu(A) = \pi(f^{-1}(A))$, then μ is Binomial with parameters n and $1/2$. (Here $f^{-1}(A) = \{x \in V : f(x) \in A\}$.)

Exercise 10 (Birth-and-Death Chains)**.** A birth-and-death chain can model the size of some population of organisms. Fix a positive integer k. Consider the state space $\Omega = \{0, 1, 2, \ldots, k\}$. The current state is the current size of the population, and at each step the size can increase or decrease by at most 1. We define $\{(p_n, r_n, q_n)\}_{n=0}^k$ such that $p_n + r_n + q_n = 1$ for each n, and

- $P(n, n+1) = p_n > 0$ for every $0 \leq n < k$.
- $P(n, n-1) = q_n > 0$ for every $0 < n \leq k$.
- $P(n, n) = r_n \geq 0$ for every $0 \leq n \leq k$.
- $q_0 = p_k = 0$.

Show that the birth-and-death chain is reversible.

Exercise 11. Give an explicit example of a Markov chain where every state has period 100.