Please provide complete and well-written solutions to the following exercises.

Due October 13, in the discussion section.

Homework 2

Exercise 1. Let P, Q be stochastic matrices of the same size. Show that PQ is a stochastic matrix. Conclude that, if r is a positive integer, then P^r is a stochastic matrix.

Exercise 2. Let A, B be events in a sample space. Let $C_1, \ldots, C_n$ be events such that $C_i \cap C_j = \emptyset$ for any $i, j \in \{1, \ldots, n\}$, and such that $\cup_{i=1}^n C_i$ is the whole sample space. Show:

$$\mathbf{P}(A|B) = \sum_{i=1}^n \mathbf{P}(A|B,\, C_i)\mathbf{P}(C_i|B).$$

(Hint: consider using the Total Probability Theorem and that $\mathbf{P}(\cdot|B)$ is a probability law.)

Exercise 3. Let $0 < p, q < 1$. Let $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$. Find the (left) eigenvectors of P, and find the eigenvalues of P. By writing any row vector $x \in \mathbf{R}^2$ as a linear combination of eigenvectors of P (whenever possible), find an expression for xP^n for any $n \geq 1$. What is $\lim_{n\to\infty} xP^n$? Is it related to the vector $\pi = (q/(p+q), p/(p+q))$?

Exercise 4. Let $G = (V, E)$ be a graph. Let $|E|$ denote the number of elements in the set E, i.e. $|E|$ is the number of edges of the graph. Prove: $\sum_{x\in V} \deg(x) = 2\,|E|$.

Exercise 5. Let A, B be events such that $B \subseteq \{X_0 = x_0\}$. Then $\mathbf{P}(A|B) = \mathbf{P}_{x_0}(A|B)$.

More generally, if A, B are events, then $\mathbf{P}_{x_0}(A|B) = \mathbf{P}(A|B, X_0 = x_0)$.

Exercise 6. Suppose we have a Markov Chain with state space Ω. Let $n \geq 0$, $\ell \geq 1$, let $x_0, \ldots, x_n \in \Omega$ and let $A \subseteq \Omega^\ell$. Using the (usual) Markov property, show that

$$\begin{aligned}\mathbf{P}((X_{n+1}, \ldots, X_{n+\ell}) \in A \,|\, (X_0, \ldots, X_n) = (x_0, \ldots, x_n)) \\ = \mathbf{P}((X_{n+1}, \ldots, X_{n+\ell}) \in A \,|\, X_n = x_n).\end{aligned}$$

Then, show that

$$\mathbf{P}((X_{n+1}, \ldots, X_{n+\ell}) \in A \,|\, X_n = x_n) = \mathbf{P}((X_1, \ldots, X_\ell) \in A \,|\, X_0 = x_n).$$

(Hint: it may be helpful to use the Multiplication Rule.)

Exercise 7. Suppose we have a Markov chain $X_0, X_1, \ldots$ with finite state space Ω. Let $y \in \Omega$. Define $L_y := \max\{n \geq 0 \colon X_n = y\}$. Is L_y a stopping time? Prove your assertion.

Exercise 8. Let x, y be points in the state space of a finite Markov Chain $(X_0, X_1, \ldots)$. Let $T_y = \min\{n \geq 1 \colon X_n = y\}$ be the first arrival time of y. Let j, k be positive integers. Show that

$$\mathbf{P}_x(T_y > kj \,|\, T_y > (k-1)j) \leq \max_{z\in\Omega} \mathbf{P}_z(T_y > j).$$

(Hint: use Exercise 6)

Exercise 9. Let x, y be points in the state space of a finite Markov Chain $(X_0, X_1, \dots)$ with transition matrix P. Let $T_y = \min\{n \geq 1 \colon X_n = y\}$ be the first arrival time of y. Let j be a positive integer. Show that

$$P^j(x, y) \leq \mathbf{P}_x(T_y \leq j).$$

(Hint: can you induct on j?)

Exercise 10. Let x, y be any states in a finite irreducible Markov chain. Show that $\mathbf{E}_x T_y < \infty$. In particular, $\mathbf{P}_y(T_y < \infty) = 1$, so all states are recurrent.