171 Final Solutions, Fall 2016[1]

1. Question 1

True/False

(a) Let $\{N(s)\}_{s\geq 0}$ be a Poisson Process with parameter $\lambda = 1$. Then

$$N(4) - N(3), N(3) - N(2), N(2) - N(1), N(1)$$

are all independent random variables.

TRUE. This is the independent increment property, Theorem 5.11 in the notes.

(b) Let $\{N(s)\}_{s\geq 0}$ be a renewal process. Then $N(1)$ and $N(0)$ are independent random variables.

FALSE. Let $\tau_1, \tau_2, \ldots$ be independent random variables so that $\mathbf{P}(\tau_i = 1) = \mathbf{P}(\tau_i = 0) = 1/2$ for all $i \geq 1$. Then $N(1) = \max\{n \geq 0 \colon T_n \leq 1\}$, $N(0) = \max\{n \geq 0 \colon T_n \leq 0\}$,

$$\mathbf{P}(N(1) = 1, N(0) = 0) = \mathbf{P}(\tau_1 = 1, \tau_2 = 1) = 1/4.$$

$$\mathbf{P}(N(1) = 1) = \mathbf{P}(\tau_1 = 1, \tau_2 = 1) = 1/4, \qquad \mathbf{P}(N(0) = 0) = \mathbf{P}(\tau_1 = 1) = 1/2$$

So, $\mathbf{P}(N(1) = 1, N(0) = 0) \neq \mathbf{P}(N(1) = 1)\mathbf{P}(N(0) = 0)$, so $N(1)$ and $N(0)$ are not independent.

(c) Suppose we have a renewal process $\{N(s)\}_{s\geq 0}$ with arrival increments $\tau_1, \tau_2, \ldots$. Let $\mu := \mathbb{E}\tau_1$. Assume that $0 < \mu < \infty$. Then

$$\mathbf{P}\left(\lim_{s\to\infty} \frac{N(s)}{s} = \frac{1}{\mu}\right) = 1.$$

TRUE. This is the Law of Large Numbers for renewal processes, Theorem 6.3 from the notes.

(d) Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Let $0 < s < t$. Then

$$\mathbb{E}B(s)B(t) = t.$$

FALSE. $\mathbb{E}B(s)B(t) = s$. Using that $B(s)$ has variance s, and using the independent increment property,

$$\begin{aligned}\mathbb{E}B(s)B(t) &= \mathbb{E}B(s)(B(t) - B(s) + B(s)) = \mathbb{E}(B(s))^2 + \mathbb{E}B(s)(B(t) - B(s))\\ &= s + (\mathbb{E}B(s))(\mathbb{E}B(t) - B(s)) = s.\end{aligned}$$

(e) Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Let $a, b > 0$. Let $T_a := \inf\{t \geq 0 \colon B(t) = a\}$. Then

$$\mathbf{P}(T_a < T_{-b}) = \frac{b}{a+b}$$

TRUE. This was Proposition 7.11 from the notes. Let $c := \mathbf{P}(T_a < T_{-b})$. Let $T := \inf\{t \geq 0 \colon B(t) \in \{a, b\}\}$. From the Optional Stopping Theorem (for continuous-time martingales) (noting that $|B(t \wedge T)| \leq \max(a, b)$ for all $t \geq 0$)

$$0 = \mathbb{E}B(0) = \mathbb{E}B(T) = ac - b(1 - c).$$

Solving for c proves the result.

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[1]March 22, 2017,

2. Question 2

Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X\colon [0,1] \to \mathbb{R}$ be a random variable such that $X(t) = t^3$ for all $t \in [0,1]$. Let

$$\mathcal{A} = \{[0,1/4), [1/4,1/2), [1/2,3/4), [3/4,1]\}$$

Compute explicitly the function $\mathbb{E}(X|\mathcal{A})$.

Solution. By definition, if $t \in [0,1/4)$, then

$$\mathbb{E}(X|\mathcal{A})(t) = \mathbb{E}(X1_{[0,1/4)})/\mathbf{P}[0,1/4) = 4\int_0^{1/4} s^3 ds = (1/4)^4 = \frac{1}{256}.$$

Similarly,

$$\mathbb{E}(X|\mathcal{A})(t) = \begin{cases} 4\int_0^{1/4} s^3 ds = (1/4)^4 = \frac{1}{256}, & \text{if}\, t \in [0,1/4) \\ 4\int_{1/4}^{1/2} s^3 ds = [(1/2)^4 - (1/4)^4] = \frac{15}{256}, & \text{if}\, t \in [1/4,1/2) \\ 4\int_{1/2}^{3/4} s^3 ds = [(3/4)^4 - (1/2)^4] = \frac{65}{256}, & \text{if}\, t \in [1/2,3/4) \\ 4\int_{3/4}^{1} s^3 ds = [1^4 - (3/4)^4] = \frac{175}{256}, & \text{if}\, t \in [3,4/1] \end{cases}$$

3. Question 3

Give an example of a martingale that is a Markov chain.

Solution. Here is one of many examples. Let $X_0, X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2$ for every $n \geq 0$. For any $n \geq 0$ define $Y_n = X_0 + \cdots + X_n$. Then $\mathbb{E}(Y_{n+1} - Y_n \,|X_n = x_n, \ldots, X_0 = X_0, Y_0 = y_0) = \mathbb{E}X_{n+1} = 0$, for any $y_0, x_0, \ldots, x_n \in \mathbb{Z}$. So, $Y_0, Y_1, \ldots$ is a martingale with respect to $X_0, X_1, \ldots$. And $Y_0, Y_1, \ldots$ is a Markov chain, since, for any $y, y_0, \ldots, y_n \in \mathbb{Z}$,

$$\begin{aligned} \mathbf{P}(Y_{n+1} = y_{n+1} \,|\, Y_n = y_n, \ldots, Y_0 = y_0) &= \mathbf{P}(X_{n+1} + Y_n = y_{n+1} \,|\, Y_n = y_n, \ldots, Y_0 = y_0) \\ &= \mathbf{P}(X_{n+1} = y_{n+1} - y_n) = \mathbf{P}(X_1 = y_{n+1} - y_n) = \mathbf{P}(X_1 + Y_0 = y_{n+1} \,|\, Y_0 = y_n) \\ &= \mathbf{P}(Y_1 = y_{n+1} \,|\, Y_0 = y_n). \end{aligned}$$

4. Question 4

Give an example of a martingale that is **not** a Markov chain.

(Your example should be a discrete time stochastic process $Y_0, Y_1, Y_2, \ldots$.)

Solution. Here is one of many examples. We will take a Markov chain and "slow it down" so that each step of the Markov chain takes two values of n to be "completed."

Let $Y_0, Y_2, Y_4, \ldots$ be independent identically distributed random variables with $\mathbf{P}(Y_n = 1) = \mathbf{P}(Y_n = -1) = 1/2$ for every $n \geq 0$. For any $n \geq 0$ even, define $X_n := Y_n + Y_{n-2} + \cdots + Y_2 + Y_0$ and for any $n \geq 1$ odd, let $X_n := X_{n-1}$. Then $X_n - X_{n-1} = 0$ for any $n \geq 1$ odd, and $\mathbb{E}(X_n - X_{n-1} \,|Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0) = (\mathbb{E}Y_{n+1}) = 0$, for any $n \geq 2$ even and for any $y_0, \ldots, y_n \in \mathbb{Z}$. So, $X_0, X_1, \ldots$ is a martingale with respect to $Y_0, Y_1, \ldots$.

However, $X_0, X_1, \ldots$ is not a Markov chain, since $\mathbf{P}(X_1 = 1|X_0 = 1) = 1$, but $\mathbf{P}(X_2 = 1|X_1 = 1) = \mathbf{P}(X_2 = 1|X_0 = 1) = 1/2$, both by definition of $X_0, X_1, \ldots$. (By the definition of a Markov chain, we should have $\mathbf{P}(X_1 = 1|X_0 = 1) = \mathbf{P}(X_2 = 1|X_1 = 1)$)

5. Question 5

Let $X \geq 0$ be a random variable such that $\mathbf{P}(X > 0) > 0$. Show that

$$\mathbb{E}(X \,|\, X > 0) \leq \frac{\mathbb{E}X^2}{\mathbb{E}X}.$$

(Hint: you can freely use the Cauchy-Schwarz inequality: $(\mathbb{E}XY)^2 \leq \mathbb{E}X^2\mathbb{E}Y^2$.)

Solution. Since $X \geq 0$ and $\mathbf{P}(X > 0) > 0$, we know that $\mathbb{E}X > 0$. So, we are required to show that $\mathbb{E}X\mathbb{E}(X|X > 0) \leq \mathbb{E}X^2$. Since $X = X \cdot 1_{\{X>0\}}$, we are required to show that $[\mathbb{E}(X \cdot 1_{\{X>0\}})]^2/\mathbf{P}(X > 0) \leq \mathbb{E}X^2$. Rearranging, we need to show that $[\mathbb{E}(X \cdot 1_{\{X>0\}})]^2 \leq \mathbb{E}X^2\mathbf{P}(X > 0)$. Since $\mathbb{E}1^2_{\{X>0\}} = \mathbb{E}1_{\{X>0\}} = \mathbf{P}(X > 0)$, our desired inequality follows from the Cauchy-Schwarz inequality.

6. Question 6

Suppose you run a (busy) car wash, and the number of cars that come to the car wash between time 0 and time $s > 0$ is a Poisson process with rate $\lambda = 1$. Suppose every car has either one, two, three, or four people in it. The probability that a car has one, two, three or four people in it is 1/4, 1/2, 1/12 and 1/6, respectively. What is the average number of cars with four people that have arrived by time $s = 60$?

Solution. From Theorem 5.17 in the notes, the number of cars with four people in it is a Poisson process with rate $\lambda \cdot (1/6) = 1/6$. So, the average number of cars with four people is the expected value $\mathbb{E}N(60)$ of a Poisson Process with rate 1/6. From Lemma 5.5 in the notes, $N(60)$ is a Poisson random variable with parameter $60(1/6) = 10$. That is, $\mathbf{P}(N(60) = n) = e^{-10}10^n/n!$ for any nonnegative integer n. So,

$$\mathbb{E}N(60) = e^{-10}\sum_{n=0}^{\infty} n\frac{10^n}{n!} = e^{-10}10\sum_{n=0}^{\infty}\frac{10^n}{n!} = e^{-10}e^{10}10 = 10.$$

7. Question 7

Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots, Z_1, Z_2, \ldots$ be random variables. Let $a, b \in \mathbb{R}$.

Assume that $X_n \leq Y_n \leq Z_n$ for any $n \geq 1$. Assume that $\mathbf{P}(\lim_{n\to\infty} X_n = a) = 1$ and $\mathbf{P}(\lim_{n\to\infty} Z_n = a) = 1$. Prove that $\mathbf{P}(\lim_{n\to\infty} Y_n = a) = 1$.

Solution. Let $C := \{\lim_{n\to\infty} X_n = a\} \cap \{\lim_{n\to\infty} Z_n = a\}$. Note that

$$\mathbf{P}(C^c) = \mathbf{P}(\{\lim_{n\to\infty} X_n \neq a\} \cup \{\lim_{n\to\infty} Z_n \neq a\}) \leq \mathbf{P}(\{\lim_{n\to\infty} X_n \neq a\}) + \mathbf{P}(\{\lim_{n\to\infty} Z_n \neq a\}) = 0$$

So, $\mathbf{P}(C) = 1$. If $\omega \in C$, then $\lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} Z_n(\omega) = a$. It follows from the Squeeze Theorem from Calculus that $\lim_{n\to\infty} Y_n(\omega) = a$, since $X_n \leq Y_n \leq Z_n$. Therefore, $\mathbf{P}(\lim_{n\to\infty} Y_n = a) \geq \mathbf{P}(C) = 1$, so $\mathbf{P}(\lim_{n\to\infty} Y_n = a) = 1$.

(If $\lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} Z_n(\omega) = a$, then for all $\varepsilon > 0$, there exists $n = n(\varepsilon)$ such that, for all $m \geq n$, $|X_m(\omega) - a| < \varepsilon$ and $|Z_m(\omega) - a| < \varepsilon$. Since $X_n(\omega) \leq Y_n(\omega) \leq Z_n(\omega)$, we have $X_n(\omega) - a \leq Y_n(\omega) - a \leq Z_n(\omega) - a$ and $a - X_n(\omega) \geq a - Y_n(\omega) \geq a - Z_n(\omega)$. So, $|Y_n(\omega) - a| \leq \max(|X_m(\omega) - a|, |Z_m(\omega) - a|) < \varepsilon$. That is, for all $\varepsilon > 0$, there exists $n = n(\varepsilon)$ such that, for all $m \geq n$, we have $|Y_n(\omega) - a| < \varepsilon$. That is, $\lim_{n\to\infty} Y_n(\omega) = a$.)

8. Question 8

Let $A := \{1+1, 1+1/2, 1+1/3, 1+1/4, \ldots\}$. Find $\inf(A)$, the greatest lower bound of A.

Let $B := \{1-1, 1-1/2, 1-1/3, 1-1/4, 1-1/5, \ldots\}$. Find $\inf(B)$.

Solution. $\inf(A) = 1$. Since every element of $a \in A$ is of the form $a = 1 + 1/n$, $n \geq 1$, we always have $a \geq 1$. So, 1 is a lower bound for A. And 1 is also the greatest lower bound for A, since for any real number $x > 1$, there exists $n \geq 1$ such that $x > 1 + 1/n$, by the Archimedean property of the real numbers. (Since $x - 1 > 0$, $1/(x-1) > 0$, and there exists an integer n such that $n > 1/(x-1)$, so that $0 < 1/n < x - 1$, i.e. $1 < 1 + 1/n < x$.)

$\inf(B) = 0$. Since every element of $b \in B$ is of the form $b = 1 - 1/n$, $n \geq 1$, we always have $b \geq 0$. So, 0 is a lower bound for B. And 0 is also the greatest lower bound for B, since for any real number $x > 0$, satisfies $x > 1 - 1 = 0$.

9. Question 9

Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion (so that $B(0) = 0$). For any $x > 0$, let $T_x := \inf\{t \geq 0 \colon B(t) = x\}$.

- Show the bound $\mathbf{P}(-x < B(t) < x) \geq \frac{x}{20\sqrt{t}}$ holds for all $t > x^2$.
- Show that $\mathbb{E}T_x = \infty$. (Hint: use a reflection principle.)

Solution. Let $x > 0$ and let $t > 0$. Since $B(t)$ is a Brownian motion, $B(t)$ has density $e^{-y^2/(2t)}\frac{1}{\sqrt{2\pi t}}$. If $t > x^2$, and if $y \in [-x, x]$, then $t > y^2$, $y^2/t < 1$ and $-y^2/(2t) > -1/2$. So,

$$\mathbf{P}(-x < B(t) < x) = \int_{-x}^{x} e^{-x^2/(2t)}\frac{1}{\sqrt{2\pi t}} \geq e^{-1/2}\int_{-x}^{x} dy\frac{1}{\sqrt{2\pi t}} = 2xe^{-1/2}(2\pi t)^{-1/2} \geq \frac{x}{20\sqrt{t}}.$$

Now, from the Reflection principle, Proposition 7.15 in the notes,

$$\mathbf{P}(T_x > t) = \mathbf{P}(-x < B(t) < x) \geq \frac{x}{20\sqrt{t}}.$$

So, $\mathbb{E}T_x = \int_0^\infty \mathbf{P}(T_x > t)dt \geq \frac{x}{20}\int_{x^2}^\infty t^{-1/2}dt = \infty$.

10. Question 10

Let $x_1, \ldots, x_n \in \mathbb{R}$, and let $t_n > \cdots > t_1 > 0$. Show that the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

has a multivariate normal distribution. That is, the joint density of $(B(t_1), \ldots, B(t_n))$ is

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1)f_{t_2-t_1}(x_2 - x_1)\cdots f_{t_n-t_{n-1}}(x_n - x_{n-1})$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/(2t)}, \qquad \forall x \in \mathbb{R},\ t > 0.$$

Solution.

$$\begin{aligned}&\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}\\ &\qquad = \{B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1, \ldots, B(t_n) - B(t_{n-1}) = x_n - x_{n-1}\}.\end{aligned}$$

The random variables listed on the right are all independent, by the independent increment property (i) of Brownian motion. So, the joint density of $(B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1}))$ is the product of the respective densities of the random variables. By property

(ii) of Brownian motion, $B(s) - B(t)$ is a Gaussian random variable with mean zero and variance $t - s$. So, the joint density of $(B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1}))$ has density $f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$. The proof is complete.