171 Midterm 2 Solutions, Winter 2017[1]

1. Question 1

Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the uniform probability law on Ω. Let $X\colon [0,1] \to \mathbb{R}$ be a random variable such that $X(t) = t^3$ for all $t \in [0,1]$. Let

$$\mathcal{A} = \{[0,1/4), [1/4,1/2), [1/2,3/4), [3/4,1]\}$$

Compute explicitly the function $\mathbb{E}(X|\mathcal{A})$.

Solution. By definition, if $t \in [0,1/4)$, then

$$\mathbb{E}(X|\mathcal{A})(t) = \mathbb{E}(X 1_{[0,1/4)})/\mathbf{P}[0,1/4) = 4\int_0^{1/4} s^3 ds = (1/4)^4 = \frac{1}{256}.$$

Similarly,

$$\mathbb{E}(X|\mathcal{A})(t) = \begin{cases} 4\int_0^{1/4} s^3 ds = (1/4)^4 = \frac{1}{256}, & \text{if}\, t \in [0,1/4) \\ 4\int_{1/4}^{1/2} s^3 ds = [(1/2)^4 - (1/4)^4] = \frac{15}{256}, & \text{if}\, t \in [1/4,1/2) \\ 4\int_{1/2}^{3/4} s^3 ds = [(3/4)^4 - (1/2)^4] = \frac{65}{256}, & \text{if}\, t \in [1/2,3/4) \\ 4\int_{3/4}^{1} s^3 ds = [1^4 - (3/4)^4] = \frac{175}{256}, & \text{if}\, t \in [3,4/1] \end{cases}$$

2. Question 2

Suppose we have a finite, irreducible, aperiodic Markov chain with transition matrix P. Since there exists a unique stationary distribution for this Markov chain, we know that one eigenvalue of P is 1. Show that any other eigenvalue λ of P satisfies $|\lambda| < 1$. (Hint: use the Convergence Theorem.)

Solution. We argue by contradiction. Suppose P has an eigenvalue λ with $|\lambda| \geq 1$ and $\lambda \neq 1$. Let μ be a (left) eigenvector of P with eigenvalue λ. Then $\mu P = \lambda\mu$, so that $\mu P^n = \lambda^n \mu$ for any $n \geq 1$. The Convergence Theorem (Theorem 3.62 in the notes) implies that as $n \to \infty$, P^n converges to a matrix Π each of whose rows is the stationary distribution π. So,

$$\mu\Pi = \lim_{n\to\infty} \mu P^n = \lim_{n\to\infty} \lambda^n \mu.$$

In particular, the limit on the right exists. This limit can only exist if $|\lambda| \leq 1$ and $\lambda \neq -1$. Since $\lambda \neq 1$, we conclude that $|\lambda| < 1$.

3. Question 3

Give an example of a martingale that is **not** a Markov chain.

(Your example should be a discrete time stochastic process $Y_0, Y_1, Y_2, \ldots$.)

Solution. Here is one of many examples. We will take a Markov chain and "slow it down" so that each step of the Markov chain takes two values of n to be "completed."

Let $Y_0, Y_2, Y_4, \ldots$ be independent identically distributed random variables with $\mathbf{P}(Y_n = 1) = \mathbf{P}(Y_n = -1) = 1/2$ for every $n \geq 0$. For any $n \geq 0$ even, define $X_n := Y_n + Y_{n-2} + \cdots + Y_2 + Y_0$ and for any $n \geq 1$ odd, let $X_n := X_{n-1}$. Then $X_n - X_{n-1} = 0$ for any $n \geq 1$ odd, and $\mathbb{E}(X_n - X_{n-1}\,|Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0) = (\mathbb{E}Y_{n+1}) = 0$, for any $n \geq 2$ even and for any $y_0, \ldots, y_n \in \mathbb{Z}$. So, $X_0, X_1, \ldots$ is a martingale with respect to $Y_0, Y_1, \ldots$.

[1] March 3, 2017,

However, $X_0, X_1, \ldots$ is not a Markov chain, since $\mathbf{P}(X_1 = 1 | X_0 = 1) = 1$, but $\mathbf{P}(X_2 = 1 | X_1 = 1) = \mathbf{P}(X_2 = 1 | X_0 = 1) = 1/2$, both by definition of $X_0, X_1, \ldots$. (By the definition of a Markov chain, we should have $\mathbf{P}(X_1 = 1 | X_0 = 1) = \mathbf{P}(X_2 = 1 | X_1 = 1)$)

4. Question 4

For the simple random walk on $\mathbb{Z}$, show that $\mathbb{E}_1 T_0 = \infty$.

Solution. From Lemma 3.69 in the notes, $\mathbf{P}_1(T_0 > r) = \mathbf{P}_0(-1 < X_r \leq 1) \geq \mathbf{P}_0(X_r = 0)$. From an Exercise, $\lim_{n\to\infty} \sqrt{2n}\,\mathbf{P}_0(X_{2n} = 0) = \sqrt{\frac{2}{\pi}}$. That is, there exists $m \geq 0$ and a constant $c > 0$ such that, for all $n \geq m$, $\mathbf{P}_0(X_{2n} = 0) \geq cn^{-1/2}$. Combining our inequalities,

$$\mathbf{P}_1(T_0 > 2n) \geq cn^{-1/2}.$$

Therefore,

$$\mathbb{E}_1 T_0 = \int_0^\infty \mathbf{P}_1(T_0 > r)dr = \sum_{n=1}^\infty \mathbf{P}_1(T_0 \geq n) \geq \sum_{n=1}^\infty \mathbf{P}_1(T_2 > 2n) \geq \sum_{n=1}^\infty cn^{-1/2} = \infty.$$

5. Question 5

Let $X_0 = 0$, and let $a < 0 < b$ be integers. Let $X_1, X_2, \ldots$ be independent identically distributed random variables so that $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n := X_0 + \cdots + X_n$. Define $T := \min\{n \geq 1 \colon Y_n \notin (a, b)\}$. First, show that $\mathbf{P}(Y_T = a) = -b/(a - b)$. Then, compute $\mathbb{E}T$.

(Hint: use martingales, somehow. And you are allowed to apply the Optional Stopping Theorem without verifying its assumptions.)

Solution. The random variables $Y_0, Y_1, \ldots$ are a martingale with respect to $X_0, X_1, \ldots$ (as proven in the notes), so the Optional Stopping Theorem (Version 2) says $\mathbb{E}(Y_T) = \mathbb{E}Y_0 = 0$, so $0 = ca + (1 - c)b$ where $c = \mathbf{P}(Y_T = a)$. Solving for c, we get $c = -b/(a - b)$.

We now claim that $\mathbb{E}T = -ab$. To see this, we use that $Y_n^2 - n$ is a martingale with respect to $X_0, X_1, \ldots$ and the Optional Stopping Theorem to get $0 = \mathbb{E}(Y_T^2 - T)$, then using $\mathbf{P}(Y_T = a) = -b/(a - b)$,

$$\begin{aligned}\mathbb{E}T = \mathbb{E}Y_T^2 &= a^2\mathbf{P}(Y_T = a) + b^2\mathbf{P}(Y_T = b)\\ &= a^2\frac{b}{b-a} + b^2\frac{(-a)}{b-a} = ab\frac{a-b}{b-a} = -ab.\end{aligned}$$

Finally, $Y_n^2 - n$ is a martingale, since

$$\begin{aligned}&\mathbb{E}(Y_{n+1}^2 - (n+1) - [Y_n^2 - n] \,|\, X_n = x_n, \ldots, X_0 = x_0, Y_0^2 = m_0)\\ &= \mathbb{E}((X_{n+1} + x_n + \cdots + x_0)^2 - (x_n + \cdots + x_0)^2 - 1)\\ &= \mathbb{E}(X_{n+1}^2 - 1) + \mathbb{E}(X_{n+1})(x_n + \cdots + x_0) = 0 + 0 = 0.\end{aligned}$$