171 Midterm 2 Solutions, Fall 2016[1]

1. Question 1

True/False

(a) Every Markov chain has at most one stationary distribution.

FALSE. Consider the Markov chain with transition matrix $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\pi = (1,0)$ and $\pi = (0,1)$ are both distinct stationary distributions for P, since $\pi = \pi P$.

(b) Let P be a transition matrix for a finite Markov chain on a state space Ω such that $P(x,y) = P(y,x)$ for all $x, y \in \Omega$. Then this Markov chain is reversible.

TRUE. Define $\pi(x) = 1/|\Omega|$ for all $x \in \Omega$. Then the reversibility condition holds.

(c) Let P be the transition matrix of a finite, irreducible Markov chain, with state space Ω and with (unique) stationary distribution π. Then there exist constants $\alpha \in (0,1)$ and $C > 0$ such that

$$\max_{x \in \Omega} \|P^n(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} \leq C\alpha^n, \qquad \forall\, n \geq 1.$$

FALSE. If the Markov chain is not aperiodic, this can be false. Suppose $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\pi = (1/2, 1/2)$ and $P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so if n is even, then for any x in the state space $\{1,2\}$, we have $\|P^n(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} = \|(1,0) - (1/2,1/2)\|_{\mathrm{TV}} \geq 1/2$, using $A = \{1\}$ in the definition of total variation distance.

(d) Every irreducible Markov chain has a stationary distribution. (A stationary distribution π for a countable Markov chain Ω satisfies $\sum_{x \in \Omega} \pi(x) = 1$, $\pi(x) \geq 0$ and $\pi(x) = \sum_{y \in \Omega} \pi(y) P(y,x)$, for all $x \in \Omega$, where P is the transition matrix of the Markov chain)

FALSE. The simple random walk on the integers has no stationary distribution. If it did have a stationary distribution, then $\pi(z) = \frac{1}{2}(\pi(z+1) + \pi(z-1))$ for every $z \in \mathbb{Z}$. Let $y \in \mathbb{Z}$ such that $\pi(y) = \max_{z \in \mathbb{Z}} \pi(z)$. (A set of nonnegative numbers summing to 1 must have a maximum element.) Then by stationarity and the definition of y, we have $\pi(y) = \frac{1}{2}(\pi(y+1) + \pi(y-1)) \leq \frac{1}{2}(\pi(y) + \pi(y)) = \pi(y)$. That is, $\pi(y) = \pi(y+1) = \pi(y-1)$. Similarly, $\pi(z) = \pi(y)$ for every $y \in \mathbb{Z}$. But then $\sum_{z \in \Omega} \pi(z) = 0$ or ∞. In either case, this is a contradiction.

(e) Let $M_0 = 0$ and let $M_0, M_1, \ldots$ be a martingale. Let T be a stopping time for the martingale. Then $\mathbb{E}M_T = 0$.

FALSE. Consider the simple random walk on the integers, and let $T := \min\{n \geq 1 \colon M_n = 1\}$. Then $M_T = 1$ so $\mathbb{E}M_T = 1 \neq 0$.

2. Question 2

Consider a finite state Markov chain with state space Ω satisfying $P(x,y) > 0$ for all $x, y \in \Omega$ with $x \neq y$. Show that the stationary distribution of the Markov chain satisfies the detailed balance condition if and only if

$$P(x,y)P(y,z)P(z,x) = P(x,z)P(z,y)P(y,x)$$

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for all $x, y, z \in \Omega$. (Hint: for the reverse implication, fix $z \in \Omega$ and define $\mu\colon \Omega \to \mathbb{R}$ so that $\mu(z) = 1$ and $\mu(y) = \frac{P(z,y)}{P(y,z)}$ for all $y \in \Omega$, $y \neq z$.)

Solution Suppose the detailed balance condition is satisfied. Since $P(x, y) > 0$ for all $x, y \in \Omega$ with $x \neq y$, the Markov chain is irreducible. So, there exists a unique stationary distribution π by Theorem 3.36 in the notes. Moreover, $\pi(x) > 0$ for every $x \in \Omega$, by Theorem 3.33 in the notes. So, if $x \in \Omega$, we repeatedly apply the detailed balance condition to get

$$\begin{aligned}\pi(x)P(x,y)P(y,z)P(z,x) &= P(y,x)\pi(y)P(y,z)P(z,x) \\ &= P(y,x)P(z,y)\pi(z)P(z,x) = P(y,x)P(z,y)P(x,z)\pi(x).\end{aligned}$$

Dividing by $\pi(x)$ completes the forward implication.

Now assume that $P(x,y)P(y,z)P(z,x) = P(x,z)P(z,y)P(y,x)$ for all $x, y, z \in \Omega$. Fix $x \in \Omega$ and define $\mu(y)$ as above. Then $P(x,y)\mu(x) = P(y,x)\mu(y)$. So, μ is reversible. So, if we define $\nu(x) := \mu(x)/\sum_{y\in\Omega}\mu(y)$ for any $x \in \Omega$, then ν is a reversible probability distribution. Proposition 3.46 from the notes implies that ν is stationary. Uniqueness of the stationary distribution (Theorem 3.36) therefore implies that $\nu = \pi$, so π is reversible, as desired.

3. Question 3

Let $X_0 = 0$, and let $a < 0 < b$ be integers. Let $X_1, X_2, \ldots$ be independent identically distributed random variables so that $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n := X_0 + \cdots + X_n$. Define $T := \min\{n \geq 1\colon Y_n \notin (a, b)\}$. First, show that $\mathbf{P}(Y_T = a) = -b/(a - b)$. Then, compute $\mathbb{E}T$. (Hint: use martingales, somehow.)

Solution. The random variables $Y_0, Y_1, \ldots$ are a martingale with respect to $X_0, X_1, \ldots$, so the Optional Stopping Theorem says $\mathbb{E}(Y_T) = \mathbb{E}Y_0 = 0$, so $0 = ca + (1 - c)b$ where $c = \mathbf{P}(Y_T = a)$. Solving for c, we get $c = -b/(a - b)$. (Note that $|Y_{n\wedge T}| \leq \max(|a|, |b|)$ for all $n \geq 0$, and $\mathbf{P}(T < \infty) = 1$ by Theorem 3.66 in the notes, so the Optional Stopping Theorem, Version 2, (Theorem 4.26) applies.)

We now claim that $\mathbb{E}T = -ab$. To see this, we use that $Y_n^2 - n$ is a martingale with respect to $X_0, X_1, \ldots$ and the Optional Stopping Theorem to get $0 = \mathbb{E}(Y_T^2 - T)$, then using $\mathbf{P}(Y_T = a) = -b/(a - b)$,

$$\begin{aligned}\mathbb{E}T = \mathbb{E}Y_T^2 &= a^2\mathbf{P}(Y_T = a) + b^2\mathbf{P}(Y_T = b) \\ &= a^2\frac{b}{b-a} + b^2\frac{(-a)}{b-a} = ab\frac{a-b}{b-a} = -ab.\end{aligned}$$

(Technically, Version 2 of the Optional Stopping Theorem does not apply here, since the martingale is not bounded. Filling in the details of the above argument requires using Version 1 of the Optional Stopping Theorem, noting that $\mathbf{P}(T < \infty) = 1$ by Theorem 3.66, then letting $n \to \infty$. Since the details here are beyond this class, no one will be penalized for having difficulties filling in these details.)

Finally, $Y_n^2 - n$ is a martingale, since

$$\begin{aligned}&\mathbb{E}(Y_{n+1}^2 - (n+1) - [Y_n^2 - n] \,|\, X_n = x_n, \ldots, X_0 = x_0, Y_0^2 = m_0)\\ &= \mathbb{E}((X_{n+1} + x_n + \cdots + x_0)^2 - (x_n + \cdots + x_0)^2 - 1)\\ &= \mathbb{E}(X_{n+1}^2 - 1) + \mathbb{E}(X_{n+1})(x_n + \cdots + x_0) = 0 + 0 = 0.\end{aligned}$$

4. Question 4

For any states x, y in a (countable) Markov chain $(X_0, X_1, \ldots)$, define

$$p^{(n)}(x,y) := \mathbf{P}(X_n = y \,|\, X_0 = x), \qquad \forall\, n \geq 1.$$

Fix a state y. Let N_y be the number of times that the Markov chains returns to y. That is, N_y is the number of positive integers n such that $X_n = y$. First, show that y is transient if and only if $\mathbb{E}_y N_y < \infty$.

Now, fix two states x, y, fix $n \geq 1$ and assume that $p^{(n)}(x,y) > 0$ and $p^{(n)}(y,x) > 0$. Show that x is transient if and only if y is transient.

Solution. From Remark 2.23 in the notes, $\mathbb{E}_y N_y = \sum_{k=1}^{\infty} \mathbf{P}(N_y \geq k)$. Now, $\mathbf{P}_y(N_y \geq k) = \mathbf{P}_y(T_y^{(k)} < \infty) = \rho_{yy}^k$, by Proposition 3.21 in the notes, where $T_y^{(k)}$ is the k^{th} return time of the Markov chain. So, if y is transient, then $\mathbb{E}_y N_y = \sum_{k=1}^{\infty} \rho_{yy}^k = \rho_{yy}/(1 - \rho_{yy}) < \infty$. And if y is not transient, then $\mathbb{E}_y N_y = \infty$.

Now, assume that x is transient. From the Chapman-Kolmogorov equation, for any $n, m \geq 1$,

$$p^{(n+m+n)}(x,x) \geq p^{(n)}(x,y) p^{(m)}(y,y) p^{(n)}(y,x).$$

Summing from $m = 1$ to ∞, this equation says

$$\mathbb{E}_x N_x \geq \sum_{m=1}^{\infty} p^{(n+m+n)}(x,x) \geq p^{(n)}(x,y) \mathbb{E}_y N_y p^{(n)}(y,x).$$

So, if x is transient, then $\mathbb{E}_y N_y < \infty$, so $\mathbb{E}_y N_y < \infty$, so y is transient. Interchanging the roles of x and y, we see that if y is transient, then x is transient.