Math 171, Fall 2016, UCLA   Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ____________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 7 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
Total:	45	

Reference sheet: Below are some definitions that may be relevant.

A **(finite or countable) Markov Chain** is a stochastic process $(X_0, X_1, X_2, \ldots)$ together with a finite or countable set Ω, which is called the **state space** of the Markov Chain, and function $P \colon \Omega \times \Omega \to [0,1]$. The random variables $X_0, X_1, \ldots$ take values in the finite set Ω. P is **stochastic**, that is all of its entries are nonnegative and

$$\sum_{y\in\Omega} P(x,y) = 1, \qquad \forall\, y \in \Omega.$$

And the stochastic process satisfies the following **Markov property**: for all $x, y \in \Omega$, for any $n \geq 1$, and for all events H_{n-1} of the form $H_{n-1} = \cap_{k=0}^{n-1}\{X_k = x_k\}$, where $x_k \in \Omega$ for all $0 \leq k \leq n-1$, such that $\mathbf{P}(H_{n-1} \cap \{X_n = x\}) > 0$, we have

$$\mathbf{P}(X_{n+1} = y \,|\, H_{n-1} \cap \{X_n = x\}) = \mathbf{P}(X_{n+1} = y \,|\, X_n = x) = P(x,y).$$

Suppose we have a Markov Chain $X_0, X_1, \ldots$ with state space Ω. Let $y \in \Omega$. Define the **first return time** of y to be the following random variable: $T_y := \min\{n \geq 1 \colon X_n = y\}$. Also, define $\rho_{yy} := \mathbf{P}_y(T_y < \infty)$.

If $\rho_{yy} = 1$, we say the state $y \in \Omega$ is **recurrent**. If $\rho_{yy} < 1$, we say the state $y \in \Omega$ is **transient**. A Markov chain is **irreducible** if any state can reach any other state, with some positive probability, if the chain runs long enough.

We say that π is a **stationary distribution** if $\pi(x) \geq 0$ for every $x \in \Omega$, $\sum_{x\in\Omega} \pi(x) = 1$, and if π satisfies $\pi = \pi P$ (that is, $\pi(x) = \sum_{y\in\Omega} \pi(y)P(y,x)$ for every $x \in \Omega$.)

Let P be the transition matrix of a finite Markov chain with state space Ω. We say that the Markov chain is **reversible** if there exists a probability distribution π on Ω satisfying the following **detailed balance condition**: $\pi(x)P(x,y) = \pi(y)P(y,x), \qquad \forall\, x, y \in \Omega.$

Let μ, ν be probability distributions on a finite state space Ω. We define the **total variation distance** between μ and ν to be $||\mu - \nu||_{\mathrm{TV}} := \max_{A\subseteq\Omega} |\mu(A) - \nu(A)|$.

Let $(X_0, X_1, \ldots)$ be a real-valued stochastic process. A **real-valued martingale with respect to** $(X_0, X_1, \ldots)$ is a stochastic process $(M_0, M_1, \ldots)$ such that $\mathbf{E}\,|M_n| < \infty$ for all $n \geq 0$, and for any $m_0, x_0, \ldots, x_n \in \mathbf{R}$,

$$\mathbf{E}(M_{n+1} - M_n | X_n = x_n, \ldots, X_0 = x_0, M_0 = m_0) = 0.$$

A **stopping time** for a martingale $M_0, M_1, \ldots$ is a random variable T taking values in $0, 1, 2, \ldots, \cup\{\infty\}$ such that, for any integer $n \geq 0$, the event $\{T = n\}$ is determined by $M_0, \ldots, M_n$. More formally, for any integer $n \geq 1$, there is a set $B_n \subseteq \Omega^{n+1}$ such that $\{T = n\} = \{(M_0, \ldots, M_n) \in B_n\}$. Put another way, the indicator function $1_{\{T=n\}}$ is a function of the random variables $M_0, \ldots, M_n$.

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (3 points) Every Markov chain has at most one stationary distribution.

   TRUE FALSE (circle one)

   (b) (3 points) Let P be a transition matrix for a finite Markov chain on a state space Ω such that $P(x, y) = P(y, x)$ for all $x, y \in \Omega$. Then this Markov chain is reversible.

   TRUE FALSE (circle one)

   (c) (3 points) Let P be the transition matrix of a finite, irreducible Markov chain, with state space Ω and with (unique) stationary distribution π. Then there exist constants $\alpha \in (0, 1)$ and $C > 0$ such that

   $$\max_{x \in \Omega} ||P^n(x, \cdot) - \pi(\cdot)||_{\text{TV}} \leq C\alpha^n, \qquad \forall\, n \geq 1.$$

   TRUE FALSE (circle one)

   (d) (3 points) Every irreducible Markov chain has a stationary distribution.

   TRUE FALSE (circle one)

   (e) (3 points) Let $M_0 = 0$ and let $M_0, M_1, \ldots$ be a martingale. Let T be a stopping time for the martingale. Then $\mathbf{E}M_T = 0$.

   TRUE FALSE (circle one)

2. (10 points) Consider a finite state Markov chain with state space Ω satisfying $P(x, y) > 0$ for all $x, y \in \Omega$ with $x \neq y$. Show that the stationary distribution of the Markov chain satisfies the detailed balance condition if and only if

$$P(x, y)P(y, z)P(z, x) = P(x, z)P(z, y)P(y, x)$$

for all $x, y, z \in \Omega$. (Hint: for the reverse implication, fix $z \in \Omega$ and define $\mu \colon \Omega \to \mathbf{R}$ so that $\mu(z) = 1$ and $\mu(y) = \frac{P(z,y)}{P(y,z)}$ for all $y \in \Omega$, $y \neq z$.)

3. (10 points) Let $X_0 = 0$, and let $a < 0 < b$ be integers. Let $X_1, X_2, \ldots$ be independent identically distributed random variables so that $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n := X_0 + \cdots + X_n$. Define $T := \min\{n \geq 1 \colon Y_n \notin (a, b)\}$. First, show that $\mathbf{P}(Y_T = a) = -b/(a - b)$. Then, compute $\mathbf{E}T$. (Hint: use martingales, somehow.)

4. (10 points) For any states x, y in a (countable) Markov chain $(X_0, X_1, \ldots)$, define

$$p^{(n)}(x,y) := \mathbf{P}(X_n = y \,|\, X_0 = x), \qquad \forall\, n \geq 1.$$

Fix a state y. Let N_y be the number of times that the Markov chains returns to y. That is, N_y is the number of positive integers n such that $X_n = y$. First, show that y is transient if and only if $\mathbf{E}_y N_y < \infty$.

Now, fix two states x, y, fix $n \geq 1$ and assume that $p^{(n)}(x,y) > 0$ and $p^{(n)}(y,x) > 0$. Show that x is transient if and only if y is transient.

(Scratch paper)