

Please provide complete and well-written solutions to the following exercises.

Due February 16, in the discussion section.

Homework 5

Exercise 1. Let X be a standard Gaussian random variable. Let $t > 0$ and let n be a positive even integer. Show that

$$\mathbf{P}(X > t) \leq \frac{(n-1)(n-3)\cdots(3)(1)}{t^n}.$$

That is, the function $t \mapsto \mathbf{P}(X > t)$ decays faster than any monomial.

Exercise 2. Let X be a random variable. Let $t > 0$. Show that

$$\mathbf{P}(|X| > t) \leq \frac{\mathbf{E}X^4}{t^4}.$$

Exercise 3 (The Chernoff Bound). Let X be a random variable and let $r > 0$. Show that, for any $t > 0$,

$$\mathbf{P}(X > r) \leq e^{-tr} M_X(t).$$

Consequently, if X_1, \dots, X_n are independent random variables with the same CDF, and if $r, t > 0$,

$$\mathbf{P}\left(\frac{1}{n} \sum_{i=1}^n X_i > r\right) \leq e^{-trn} (M_{X_1}(t))^n.$$

For example, if X_1, \dots, X_n are independent Bernoulli random variables with parameter $0 < p < 1$, and if $r, t > 0$,

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_n}{n} - p > r\right) \leq e^{-trn} (e^{-tp}[pe^t + (1-p)])^n.$$

And if we choose t appropriately, then the quantity $\mathbf{P}\left(\frac{1}{n} \sum_{i=1}^n (X_i - p) > r\right)$ becomes exponentially small as either n or r become large. That is, $\frac{1}{n} \sum_{i=1}^n X_i$ becomes very close to its mean. Importantly, the Chernoff bound is much stronger than either Markov's or Chebyshev's inequality, since they only respectively imply that

$$\mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - p\right| > r\right) \leq \frac{2p(1-p)}{r}, \quad \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - p\right| > r\right) \leq \frac{p(1-p)}{nr^2}.$$

Exercise 4. Let X_1, X_2, \dots be independent random variables, each with exponential distribution with parameter $\lambda = 1$. For any $n \geq 1$, let $Y_n := \max(X_1, \dots, X_n)$. Let $0 < a < 1 < b$. Show that $\mathbf{P}(Y_n \leq a \log n) \rightarrow 0$ as $n \rightarrow \infty$, and $\mathbf{P}(Y_n \leq b \log n) \rightarrow 1$ as $n \rightarrow \infty$. Conclude that $Y_n / \log n$ converges to 1 in probability as $n \rightarrow \infty$.

Exercise 5. We say that random variables X_1, X_2, \dots converge to a random variable X in L_2 if

$$\lim_{n \rightarrow \infty} \mathbf{E} |X_n - X|^2 = 0.$$

Show that, if X_1, X_2, \dots converge to X in L_2 , then X_1, X_2, \dots converges to X in probability.

Is the converse true? Prove your assertion.

Exercise 6. Let X_1, X_2, \dots be independent, identically distributed random variables such that $\mathbf{E}|X_1| < \infty$ and $\text{var}(X_1) < \infty$. For any $n \geq 1$, define

$$Y_n := \frac{1}{n} \sum_{i=1}^n X_i^2.$$

Show that Y_1, Y_2, \dots converges in probability. Express the limit in terms of $\mathbf{E}X_1$ and $\text{var}(X_1)$.