Please provide complete and well-written solutions to the following exercises.

Due February 2, in the discussion section.

Homework 3

Exercise 1. Let X be a random variable. Assume that $M_X(t)$ exists for all $t \in \mathbf{R}$, and assume we can differentiate under the expected value any number of times. For any positive integer n, show that

$$\frac{d^n}{dt^n}|_{t=0} M_X(t) = \mathbf{E}(X^n).$$

So, in principle, all moments of X can be computed just by taking derivatives of the moment generating function.

Exercise 2. Let X be a standard Gaussian random variable. Compute an explicit formula for the moment generating function of X. (Hint: completing the square might be helpful.) From this explicit formula, compute an explicit formula for all moments of the Gaussian random variable. (The $2n^{th}$ moment of X should be something resembling a factorial.)

Exercise 3. Construct two random variables $X, Y : \Omega \to \mathbf{R}$ such that $X \neq Y$ but $M_X(t), M_Y(t)$ exist for all $t \in \mathbf{R}$, and such that $M_X(t) = M_Y(t)$ for all $t \in \mathbf{R}$.

Exercise 4. Unfortunately, there exist random variables X, Y such that $\mathbf{E}X^n = \mathbf{E}Y^n$ for all $n = 1, 2, 3, \ldots$, but such that X, Y do not have the same CDF. First, explain why this does not contradict the Lévy Continuity Theorem, Weak Form. Now, let $-1 < a < 1$, and define a density

$$f_a(x) := \begin{cases} \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\log x)^2}{2}} (1 + a \sin(2\pi \log x)) & \text{, if } x > 0 \\ 0 & \text{, otherwise.} \end{cases}$$

Suppose X_a has density f_a. If $-1 < a, b < 1$, show that $\mathbf{E}X_a^n = \mathbf{E}X_b^n$ for all $n = 1, 2, 3, \ldots$. (Hint: write out the integrals, and make a change of variables $s = \log(x) - n$.)