Please provide complete and well-written solutions to the following exercises.

Due January 26, in the discussion section.

Homework 2

Exercise 1. Let X, Y be random variables with $\mathbf{E}X^2 < \infty$ and $\mathbf{E}Y^2 < \infty$. Prove the **Cauchy-Schwarz inequality**:

$$\mathbf{E}(XY) \leq (\mathbf{E}X^2)^{1/2}(\mathbf{E}Y^2)^{1/2}.$$

Then, deduce the following when X, Y both have finite variance:

$$|\mathrm{cov}(X, Y)| \leq (\mathrm{var}(X))^{1/2}(\mathrm{var}(Y))^{1/2}.$$

(Hint: in the case that $\mathbf{E}Y^2 > 0$, expand out the product $\mathbf{E}(X - Y\mathbf{E}(XY)/\mathbf{E}Y^2)^2$.)

Exercise 2. Let X be a binomial random variable with parameters $n = 2$ and $p = 1/2$. So, $\mathbf{P}(X = 0) = 1/4$, $\mathbf{P}(X = 1) = 1/2$ and $\mathbf{P}(X = 2) = 1/4$. And X satisfies $\mathbf{E}X = 1$ and $\mathbf{E}X^2 = 3/2$.

Let Y be a geometric random variable with parameter $1/2$. So, for any positive integer k, $\mathbf{P}(Y = k) = 2^{-k}$. And Y satisfies $\mathbf{E}Y = 2$ and $\mathbf{E}Y^2 = 6$.

Let Z be a Poisson random variable with parameter 1. So, for any nonnegative integer k, $\mathbf{P}(Z = k) = \frac{1}{e}\frac{1}{k!}$. And Z satisfies $\mathbf{E}Z = 1$ and $\mathbf{E}Z^2 = 2$.

Let W be a discrete random variable such that $\mathbf{P}(W = 0) = 1/2$ and $\mathbf{P}(W = 4) = 1/2$, so that $\mathbf{E}W = 2$ and $\mathbf{E}W^2 = 8$.

Assume that X, Y, Z and W are all independent. Compute

$$\mathrm{var}(X + Y + Z + W).$$

Exercise 3. Let $X_1, \ldots, X_n$ be random variables with finite variance. Define an $n \times n$ matrix A such that $A_{ij} = \mathrm{cov}(X_i, X_j)$ for any $1 \leq i, j \leq n$. Show that the matrix A is positive semidefinite. That is, show that for any $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, we have

$$y^T A y = \sum_{i,j=1}^{n} y_i y_j A_{ij} \geq 0.$$

Exercise 4 (Another Total Expectation Theorem)**.** Using the definition of $\mathbf{E}(X|Y)$, prove the following theorem, which can be considered as a version of a Total Expectation Theorem:

$$\mathbf{E}(\mathbf{E}(X|Y)) = \mathbf{E}(X).$$

Exercise 5. If X is a random variable, and if $f(t) := \mathbf{E}(X-t)^2$, $t \in \mathbf{R}$, then the function $f\colon \mathbf{R} \to \mathbf{R}$ is uniquely minimized when $t = \mathbf{E}X$. This follows e.g. by writing

$$\mathbf{E}(X-t)^2 = \mathbf{E}(X - \mathbf{E}(X) + \mathbf{E}(X) - t)^2$$
$$= \mathbf{E}(X-\mathbf{E}(X))^2 + (\mathbf{E}X - t)^2 + 2\mathbf{E}[(X-\mathbf{E}X)(\mathbf{E}X - t)] = \mathbf{E}(X-\mathbf{E}(X))^2 + (\mathbf{E}X-t)^2.$$

So, the choice $t = \mathbf{E}X$ is the smallest, and it recovers the definition of variance, since $\mathrm{var}(X) = \mathbf{E}(X-\mathbf{E}X)^2$.

A similar minimizing property holds for conditional expectation. Let $h\colon \mathbf{R} \to \mathbf{R}$. Show that the quantity $\mathbf{E}(X-h(Y))^2$ is minimized among all functions h when $h(Y) = \mathbf{E}(X|Y)$. (Hint: Exercise 4 might be helpful.)

Exercise 6. Toys are stored in small boxes, small boxes are stored in large crates, and large crates comprise a shipment. Let X_i be the number of toys in small box $i \in \{1,2,\ldots\}$. Assume that $X_1, X_2, \ldots$ all have the same CDF. Let Y_i be the number of small boxes in large crate $i \in \{1,2,\ldots\}$. Assume that $Y_1, Y_2, \ldots$ all have the same CDF. Let Z be the number of large crates in the shipment. Assume that $X_1, X_2, \ldots, Y_1, Y_2, \ldots, Z$ are all independent, nonnegative integer-valued random variables, each with expected value 10 and variance 16.

Compute the expected value and variance of the number of toys in the shipment.

Exercise 7. Let $0 < p < 1$. Suppose you have a biased coin which has a probability p of landing heads, and probability $1-p$ of landing tails, each time it is flipped. Also, suppose you have a fair six-sided die (so each face of the cube has a distinct label from the set $\{1,2,3,4,5,6\}$, and each time you roll the die, any face of the cube is rolled with equal probability.)

Let N be the number of coin flips you need to do until the first head appears. Now, roll the fair die N times. Let S be the sum of the results of the N rolls of the die. Compute $\mathbf{E}S$ and $\mathrm{var}(S)$.

Exercise 8. Let $f\colon \mathbf{R} \to \mathbf{R}$ be twice differentiable function. Assume that f is convex. That is, $f''(x) \geq 0$, or equivalently, the graph of f lies above any of its tangent lines. That is, for any $x, y \in \mathbf{R}$,

$$f(x) \geq f(y) + f'(y)(x-y).$$

(In Calculus class, you may have referred to these functions as "concave up.") Let X be a discrete random variable. By setting $y = \mathbf{E}(X)$, prove **Jensen's inequality**:

$$\mathbf{E}f(X) \geq f(\mathbf{E}(X)).$$

In particular, choosing $f(x) = x^2$, we have $\mathbf{E}(X^2) \geq (\mathbf{E}(X))^2$.