170B Final Solutions, Fall 2017[1]

1. Question 1

True/False

(a) Let $A_1, A_2, \ldots$ be subsets of a sample space Ω. Let $\mathbf{P}$ denote a probability law on Ω. Then

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \mathbf{P}\left(\cup_{n=1}^{\infty} A_n\right)$$

FALSE. Let $A_1 = A_2 = \Omega$ and let $\emptyset = A_3 = A_4 = \cdots$. Then the left side is $1+1=2$, but the right side is $\mathbf{P}(\Omega) = 1$.

(b) Let X be a continuous random variable. Let f_X be the density function of X. Then, for any $t \in \mathbb{R}$, $\frac{d}{dt}\mathbf{P}(X \leq t)$ exists, and

$$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t).$$

FALSE. Let $f_X(t) := 1$ for any $t \in [0,1]$ and let $f_X(t) := 0$ otherwise. Then

$$\mathbf{P}(X \leq t) = \begin{cases} 0 & , \text{if}\, t < 0 \\ t & , \text{if}\, 0 \leq t \leq 1 \\ 1 & , \text{if}\, t > 1 \end{cases}.$$

In particular, $\frac{d}{dt}\mathbf{P}(X < t)$ does not exist at $t = 0$.

(c) Let X be a random variable such that $\mathbb{E}X^4 < \infty$. Then $\mathbb{E}X^2 < \infty$.

TRUE. By Jensen's inequality, $(\mathbb{E}X^2)^2 \leq \mathbb{E}X^4 < \infty$.

(d) Let X be a random variable such that $\mathbb{E}(X^6) = 16$. Then

$$\mathbf{P}(|X| > 2) \leq 1/4.$$

TRUE. By Markov's inequality,

$$\mathbf{P}(|X| > 2) = \mathbf{P}(|X|^6 > 2^6) \leq \mathbb{E}X^6/2^6 = 2^4/2^6 = 1/4.$$

(e) Let $i = \sqrt{-1}$. Let $X_1, X_2, \ldots$ be random variables such that, for any $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}e^{itX_n} = e^{-t^2/2}.$$

Then $X_1, X_2, \ldots$ converges in distribution to a standard Gaussian random variable.

TRUE. This is basically how we proved the Central Limit Theorem (Theorem 3.21 in the notes). This assertion follows by the Levy Continuity Theorem, and using that $\mathbb{E}e^{itZ} = e^{-t^2/2}$ for all $t \in \mathbb{R}$ where Z is a standard Gaussian random variable (Prop. 2.55 in the notes).

(f) Let X be a random variable with $\mathbb{E}\,|X| = 3$. Then

$$\mathbf{P}(X > t) \leq \frac{3}{t}, \quad \forall\, t \in \mathbb{R}$$

FALSE. If $t = -1$, then this says $\mathbf{P}(X > t) \leq -3$, which cannot be true.

(g) Let $\{N(s)\}_{s\geq 0}$ be a Poisson Process with parameter $\lambda = 1$. Then

$$N(4) - N(3), N(3) - N(2), N(2) - N(1), N(1)$$

are all independent random variables.

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TRUE. This is the independent increment property (recalling $N(1) = N(1) - 0 = N(1) - N(0)$.)

(h) If a sequence of random variables $X_1, X_2, \ldots$ converges in distribution to a random variable X, then $X_1, X_2, \ldots$ converges in probability to X.

FALSE. Let $\Omega = [0, 1]$. For any $n \geq 1$, let

$$X_n(\omega) := \begin{cases} (-1)^n & , \text{if } \omega \in [0, 1/2) \\ (-1)^{n+1} & , \text{if } \omega \in [1/2, 1]. \end{cases}$$

Then $X_1, X_2, \ldots$ all have the same distribution, so they converge in distribution to e.g. $X := X_1$, but they do not converge in probability; $\mathbf{P}(|X_n - X| > 1/2) = 1$ for all n even, so $\lim_{n\to\infty} \mathbf{P}(|X_n - X| > 1/2) \neq 0$.

(i) If a sequence of random variables $X_1, X_2, \ldots$ converges in distribution to a random variable X, then

$$\lim_{n\to\infty} \mathbb{E}X_n^2 = \mathbb{E}X^2$$

FALSE. Let $\Omega = [0, 1]$. For any $n \geq 1$, let

$$X_n(\omega) := \begin{cases} n & , \text{if } \omega \in [0, 1/n] \\ 0 & , \text{if } \omega \in (1/n, 1]. \end{cases}$$

Then $\mathbb{E}X_n^2 = n$ for all $n \geq 1$, but $X_1, X_2, \ldots$ converges in probability to 0 as $n \to \infty$, as shown in class. So, $\lim_{n\to\infty} \mathbb{E}X_n^2 = \infty \neq 0 = \mathbb{E}X$.

2. Question 2

Let X, Y be independent random variables. Suppose X has Fourier Transform

$$\phi_X(t) = e^{-t^2/2}, \qquad \forall\, t \in \mathbb{R}.$$

(Recall that $\phi_X(t) = \mathbb{E}e^{itX}$ where $i = \sqrt{-1}$.) Suppose Y has Fourier Transform

$$\phi_Y(t) = \cos(t), \qquad \forall\, t \in \mathbb{R}.$$

Compute $\mathbb{E}\Big[(X+Y)^2\Big]$.

Solution 1. Since X, Y are independent, we have $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = e^{-t^2/2}\cos(t)$ for all $t \in \mathbb{R}$ by Proposition 2.54 in the notes. Also recalling the proof of Exercise 2.52,

$$\frac{d^2}{dt^2}|_{t=0}\phi_{X+Y}(t) = \mathbb{E}\frac{d^2}{dt^2}|_{t=0}e^{it(X+Y)} = i^2\mathbb{E}(X+Y)^2.$$

So,

$$\begin{aligned}\mathbb{E}(X+Y)^2 = -\frac{d^2}{dt^2}|_{t=0}\phi_{X+Y}(t) &= \frac{d}{dt}|_{t=0}(t\cos(t)e^{-t^2/2} + \sin(t)e^{-t^2/2}) \\ &= -t(t\cos(t) + \sin(t))e^{-t^2} + (\cos(t) - t\sin(t) + \cos(t))e^{-t^2/2}|_{t=0} = 2.\end{aligned}$$

Solution 2. As mentioned above, and using that

$$\frac{d}{dt}|_{t=0}\phi_X(t) = \mathbb{E}\frac{d}{dt}|_{t=0}e^{itX} = i\mathbb{E}X.$$

$$\mathbb{E}X = -i\frac{d}{dt}|_{t=0}\phi_X(t) = -i\frac{d}{dt}|_{t=0}e^{-t^2/2} = [ite^{-t^2/2}]_{t=0} = 0.$$

$$\mathbb{E}X^2 = -\frac{d^2}{dt^2}|_{t=0}\phi_X(t) = -\frac{d^2}{dt^2}|_{t=0}e^{-t^2/2} = \frac{d}{dt}|_{t=0}te^{-t^2/2} = 1.$$

$$\mathbb{E}Y = -i\frac{d}{dt}|_{t=0}\phi_Y(t) = \frac{d}{dt}|_{t=0}\cos(t) = [i\sin(t)]_{t=0} = 0.$$

$$\mathbb{E}Y^2 = -\frac{d^2}{dt^2}|_{t=0}\phi_Y(t) = -\frac{d^2}{dt^2}|_{t=0}\cos(t) = \cos(0) = 1.$$

Therefore, using also that X, Y are independent,

$$\mathbb{E}(X+Y)^2 = \mathbb{E}X^2 + \mathbb{E}Y^2 + 2\mathbb{E}(XY) = 1 + 1 + (\mathbb{E}X)(\mathbb{E}Y) = 2 + 0 \cdot 0 = 2.$$

3. Question 3

Let X be a random variable uniformly distributed on $[0,1]$.

Let Y be a random variable such that $Y = X$. (Note that Y is uniformly distributed on $[0,1]$.)

Find the density of $X+Y$.

Solution. Using the definition of X and Y, we have

$$\mathbf{P}(X+Y \leq t) = \mathbf{P}(2X \leq t) = \mathbf{P}(X \leq t/2) = \begin{cases} 0 & \text{, if } t \leq 0 \\ t/2 & \text{, if } 0 < t \leq 2 \\ 1 & \text{, if } t > 2. \end{cases}$$

So,

$$f_{X+Y}(t) = \frac{d}{dt}\mathbf{P}(X+Y \leq t) = \begin{cases} 0 & \text{, if } t \leq 0 \\ 1/2 & \text{, if } 0 < t \leq 2 \\ 0 & \text{, if } t > 2. \end{cases}$$

4. Question 4

Markov's inequality says: for any random variable X with $X \geq 0$, we have

$$\mathbf{P}(X > t) \leq \frac{\mathbb{E}X}{t}, \qquad \forall\, t > 0.$$

Prove Markov's inequality.

Solution. Let $t > 0$. Let Y be a random variable such that

$$Y = \begin{cases} t & \text{, if } X \geq t \\ 0 & \text{, if } X < t. \end{cases}$$

By definition of Y, we have $Y \leq X$. Therefore, $\mathbb{E}Y \leq \mathbb{E}X$ by Exercise **??**. By the definition of Y, $\mathbb{E}Y = t\mathbf{P}(X \geq t)$. That is,

$$t\mathbf{P}(X \geq t) \leq \mathbb{E}(X).$$

5. Question 5

Using the Central Limit Theorem, prove the Weak Law of Large Numbers.

(You may assume that $X_1, X_2, \ldots$ are independent, identically distributed random variables such that $\mathbb{E}\,|X_1| < \infty$ and $0 < \mathrm{var}(X_1) < \infty$.)

Solution. Let $\varepsilon > 0$. Let $\sigma := \sqrt{\mathrm{var}(X_1)}$. Then

$$\mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}X_1\right| > \varepsilon\right) = \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{n}\right| > \varepsilon\right)$$
$$= \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{\sigma\sqrt{n}}\right| > \frac{\sqrt{n}\varepsilon}{\sigma}\right)$$

So, for any $N > 0$, there exists $m > 0$ such that, for all $n > m$, we have

$$\mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}X_1\right| > \varepsilon\right) \leq \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{\sigma\sqrt{n}}\right| > N\right).$$

(For example, choose $m = N^2\sigma^2/\varepsilon^2$, so if $n > m$, then $\sqrt{n}\varepsilon/\sigma > \sqrt{m}\varepsilon/\sigma = N$, so the set on the left is contained in the set on the right.) Letting $n \to \infty$ and using the Central Limit Theorem,

$$0 \leq \lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}X_1\right| > \varepsilon\right) \leq \lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{\sigma\sqrt{n}}\right| > N\right)$$
$$= 2\int_N^\infty e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}.$$

The left side does not depend on N, so we let $N \to \infty$ to conclude that

$$0 \leq \lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mathbb{E}X_1\right| > \varepsilon\right) \leq \lim_{N\to\infty} 2\int_N^\infty e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} = 0.$$

6. Question 6

Let $X_1, X_2, \ldots$ be a Bernoulli process with parameter $p = 1/2$. What is the expected number of trials that have to occur before we see two consecutive "successes"?

(Your final answer can be left as an infinite sum of numbers.)

Solution 1. Let T be the number of coin flips that occur until two successive heads occur. From the Total Expectation Theorem,

$$\mathbb{E}T = \mathbb{E}(T|X_1 = 0)\mathbf{P}(X_1 = 0) + \mathbb{E}(T|X_1 = 1, X_2 = 0)\mathbf{P}(X_1 = 1, X_2 = 0)$$
$$+ \mathbb{E}(T|X_1 = 1, X_2 = 1)\mathbf{P}(X_1 = 1, X_2 = 1)$$
$$= \frac{1}{2}\mathbb{E}(T|X_1 = 0) + \frac{1}{4}\mathbb{E}(T|X_1 = 1, X_2 = 0) + \frac{1}{4}\mathbb{E}(T|X_1 = 1, X_2 = 1).$$

From the fresh-start property (or Markov property) of the Bernoulli process, $X_1, X_2, \ldots$ is also a Bernoulli process. That is, if we condition on $X_1 = 0$, then $\mathbb{E}(T|X_1 = 0) = 1 + \mathbb{E}T$. Similarly, $\mathbb{E}(T|X_1 = 1, X_2 = 0) = 2 + \mathbb{E}T$. Also, $\mathbb{E}(T|X_1 = 1, X_2 = 1) = 2$, since both successes occurred during the first two coin flips in this case. In summary,

$$\mathbb{E}T = \frac{1}{2}(1 + \mathbb{E}T) + \frac{1}{4}(2 + \mathbb{E}T) + \frac{1}{4}(2).$$

Rearranging, we get

$$\frac{1}{4}\mathbb{E}T = \frac{3}{2}.$$

That is, $\mathbb{E}T = 6$.

Solution 2. Let T_1 be the number of coin flips that occur until the first success occurs. For any $i \geq 2$, let T_i be the number of coin flips that occur between the i^{th} success and the $(i-1)^{st}$ success. Then the event that two consecutive heads occurs can be written as the disjoint union

$$\cup_{j=2}^{\infty}\{T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j\}.$$

Let T be the number of coin flips that occur until two successive heads occur. Then, by the Total Expectation Theorem, we have

$$\begin{aligned}\mathbb{E}T &= \sum_{j=2}^{\infty}\mathbb{E}(T \,|\, T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j)\mathbf{P}(T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j)\\ &= \sum_{j=2}^{\infty}\mathbb{E}(\sum_{k=1}^{j} T_k \,|\, T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j)\mathbf{P}(T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j)\end{aligned}$$

From the notes, we know that $T_1, T_2, \ldots$ are independent geometric random variables with parameter $p = 1/2$. Therefore,

$$\begin{aligned}\mathbb{E}T &= \mathbb{E}T_1 + \sum_{j=2}^{\infty}\mathbb{E}(\sum_{k=2}^{j} T_k \,|\, T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j)\mathbf{P}(T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j)\\ &= \mathbb{E}T_1 + 1 + \sum_{j=3}^{\infty}\mathbb{E}(\sum_{k=2}^{j-1} T_k \,|\, T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j)\mathbf{P}(T_j = 1,\, T_i > 1,\, \forall\, 2 \leq i < j)\\ &= \mathbb{E}T_1 + 1 + \sum_{j=3}^{\infty}(j-3)\mathbb{E}(T_1 \,|\, T_1 > 1)2^{-(j-2)} = \mathbb{E}T_1 + 1 + \sum_{j=3}^{\infty}(j-3)(1+\mathbb{E}T_1)2^{-(j-2)}\\ &= \mathbb{E}T_1 + 1 + (1+\mathbb{E}T_1)\sum_{j=1}^{\infty}(j-1)2^{-j} = \mathbb{E}T_1 + 1 + (1+\mathbb{E}T_1)(\mathbb{E}T_1 - 1) = 2 + 1 + (3)(1) = 6.\end{aligned}$$

7. Question 7

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables. Assume that $\mathbb{E}X_1 = 1/2$ and $\mathrm{var}(X_1) = 3/4$.

(i) Compute

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{X_1 + \cdots + X_n}{n} > 1\right).$$

(ii) For any $n \geq 1$, define

$$Y_n := \frac{X_1^2 + \cdots + X_n^2}{X_1 + \cdots + X_n}.$$

Does $Y_1, Y_2, \ldots$ converge almost surely? If so, what does $Y_1, Y_2, \ldots$ converge to almost surely?

Solution. From the Weak Law of Large Numbers, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1+\cdots+X_n}{n} - \frac{1}{2}\right| > \varepsilon\right) = 0.$$

So, choosing $\varepsilon = 1/2$,

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \mathbf{P}\left(\frac{X_1+\cdots+X_n}{n} > 1\right) &= \lim_{n\to\infty} \mathbf{P}\left(\frac{X_1+\cdots+X_n}{n} - \frac{1}{2} > \frac{1}{2}\right) \\ &\leq \lim_{n\to\infty} \mathbf{P}\left(\left|\frac{X_1+\cdots+X_n}{n} - \frac{1}{2}\right| > \frac{1}{2}\right) = 0. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \mathbf{P}\left(\frac{X_1+\cdots+X_n}{n} > 1\right) = 0$.

We now write

$$Y_n = \frac{X_1^2+\cdots+X_n^2}{X_1+\cdots+X_n} = \frac{X_1^2+\cdots+X_n^2}{n}\frac{n}{X_1+\cdots+X_n}.$$

From the Strong Law of Large numbers, $\frac{X_1^2+\cdots+X_n^2}{n}$ converges almost surely to $\mathbb{E}X_1^2 = \mathrm{var}(X_1) + (\mathbb{E}X_1)^2 = 3/4 + 1/4 = 1$. (Note that $X_1^2, X_2^2, \ldots$ are independent, identically distributed since $X_1, X_2, \ldots$ are as well. For example, $\mathbf{P}(X_i^2 \leq t) = \mathbf{P}(X_i \leq \sqrt{t}) = \mathbf{P}(X_1 \leq \sqrt{t}) = \mathbf{P}(X_1^2 \leq t)$ for any $i \geq 1$, $t > 0$.) Also, $\frac{X_1+\cdots+X_n}{n}$ converges almost surely to $\mathbb{E}X_1 = 1/2$. That is, with probability 1, $\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} = 1/2$. So, applying limit laws, with probability 1, $\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} = 2$.

In summary, on a set $A \subseteq \Omega$ with $\mathbf{P}(A) = 1$, for all $\omega \in B$, $\lim_{n\to\infty} \frac{n}{X_1(\omega)+\cdots+X_n(\omega)} = 2$. And on a set $B \subseteq \Omega$ with $\mathbf{P}(B) = 1$, for all $\omega \in B$, $\lim_{n\to\infty} \frac{X_1^2(\omega)+\cdots+X_n^2(\omega)}{n} = 1$. Note that $\mathbf{P}(A\cap B)+\mathbf{P}(A\setminus B)+\mathbf{P}(B\setminus A)+\mathbf{P}((A\cup B)^c) = 1$ and $\mathbf{P}(A\setminus B) \leq \mathbf{P}(B^c) = 1-\mathbf{P}(B) = 0$, $\mathbf{P}(B\setminus A) \leq \mathbf{P}(A^c) = 1-\mathbf{P}(A) = 0$, and $\mathbf{P}((A\cup B)^c) \leq \mathbf{P}(B^c) = 1-\mathbf{P}(B) = 0$. Therefore, $\mathbf{P}(A\cap B) = 1$. So, on the set $A\cap B$, using the product limit law, for all $\omega \in A\cap B$, we have

$$\lim_{n\to\infty} \frac{X_1^2(\omega)+\cdots+X_n^2(\omega)}{X_1(\omega)+\cdots+X_n(\omega)} = \left(\lim_{n\to\infty} \frac{X_1^2(\omega)+\cdots+X_n^2(\omega)}{n}\right)\left(\lim_{n\to\infty} \frac{n}{X_1(\omega)+\cdots+X_n(\omega)}\right) = 1\cdot 2 = 2.$$

8. Question 8

Let $X_1, X_2, \ldots$ be a Bernoulli process with parameter $p = 1/2$. Define $N := \min\{n \geq 1 \colon X_n \neq X_1\}$. For any $n \geq 1$, define $Y_n := X_{N+n-2}$. Show that $\mathbf{P}(Y_n = 1) = 1/2$ for all $n \geq 1$, but $Y_1, Y_2, \ldots$ is not a Bernoulli process.

Solution. Since $X_1, X_2, \ldots$ are independent, identically distributed random variables, $N-1$ is a geometric random variable. Let $n \geq 3$. By the Total Probability Theorem,

$$\begin{aligned} \mathbf{P}(Y_n = 1) &= \sum_{m=1}^{\infty} \mathbf{P}(Y_n = 1 | N-1 = m)\mathbf{P}(N-1 = m) = \sum_{m=1}^{\infty} \mathbf{P}(X_{m+n-1} = 1 | N-1 = m)(1-p)^{m-1}p \\ &= \sum_{m=1}^{\infty} \mathbf{P}(X_{m+n-1} = 1 | X_2 \neq X_1, \ldots, X_{m+1} \neq X_1, X_{m+2} = X_1)(1-p)^{m-1}p \end{aligned}$$

If $n > 3$, then independence of $X_1, X_2, \ldots$ says

$$\mathbf{P}(X_{m+n-1} = 1 | X_2 \neq X_1, \ldots, X_{m+1} \neq X_1, X_{m+2} = X_1) = \mathbf{P}(X_{m+n-2} = 1) = p.$$

If $n=3$, then

$$\mathbf{P}(X_{m+n-1}=1|X_2\neq X_1,\ldots,X_{m+1}\neq X_1,X_{m+2}=X_1)=\mathbf{P}(X_{m+2}=1|X_{m+2}=X_1)=p.$$

So, for any $n\geq 3$,

$$\mathbf{P}(Y_n=1)=p\sum_{m=1}^{\infty}p(1-p)^{m-1}=p=1/2.$$

If $n=1$, then $Y_n=X_{N-1}=X_1$ by definition of N, so $\mathbf{P}(Y_n=1)=p=1/2$.

If $n=2$, then $Y_n=X_N=1-X_1$, by definition of n, so $\mathbf{P}(Y_n=1)=1-p=1/2$. Also, since $Y_1+Y_2=1$, $\mathbb{E}Y_1Y_2=\mathbb{E}Y_1(1-Y_1)=p-1\neq\mathbb{E}Y_1\mathbb{E}Y_2=p^2$, so Y_1,Y_2 are not independent, so $Y_1,Y_2,\ldots$ is not a Bernoulli process.

9. Question 9

Let X be a random variable such that $|X|\leq 1$, $X\leq 1/2$ and $\mathbb{E}X=0$.

Is it true that $\mathbb{E}(X^2)\leq 1/4$?

If this inequality is true, prove it. If this inequality is false, provide a counterexample, and justify your reasoning.

Solution. This inequality is false. Let X so that $\mathbf{P}(X=1/2)=2/3$ and $\mathbf{P}(X=-1)=1/3$. Then $|X|\leq 1$, $X\leq 1/2$, and $\mathbb{E}X=(2/3)(1/2)-(1/3)=0$. But $\mathbb{E}X^2=(1/3)(-1)^2+(2/3)(1/2)^2=1/3+1/6=1/2>1/4$.

10. Question 10

Let $X_0:=x_0\in\mathbb{Z}$. Let $X_1,X_2,\ldots$ be independent random variables such that $\mathbf{P}(X_n=1)=\mathbf{P}(X_n=-1)=1/2$ for all $n\geq 1$. Let $S_0,S_1,\ldots$ be the corresponding random walk started at x_0. Let $a,b\in\mathbb{Z}$ such that $a<x_0<b$. Let $T:=\min\{n\geq 1\colon S_n\in\{a,b\}\}$. Show:

$$\mathbf{P}(S_T=a)=\frac{x_0-b}{a-b}.$$

(You may assume that $\mathbf{P}(T<\infty)=1$.)

Solution. We claim that T is a stopping time. For any positive integer n,

$$\{T=n\}=\{S_0\in\{a,b\}^c,\ldots,S_{n-1}\in\{a,b\}^c,S_n\in\{a,b\}\}.$$

Also, $|S_{n\wedge T}|\leq\max(|a|,|b|)$, so the Optional Stopping Theorem, Version 2, applies. Let $c:=\mathbf{P}(S_T=a)$. Then

$$x_0=\mathbb{E}S_0=\mathbb{E}S_T=ac+(1-c)b.$$

Solving for c, we get

$$c=\frac{x_0-b}{a-b}.$$