Math 170B, Winter 2017, UCLA — Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ____________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 16 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 180 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	29	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	119	

Reference sheet

Below are some definitions that may be relevant.

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in probability** to a random variable X if: for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) = 0.$$

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in distribution** to a random variable X if, for any $t \in \mathbf{R}$ such that the CDF of X is continuous at t,

$$\lim_{n\to\infty} \mathbf{P}(X_n \leq t) = \mathbf{P}(X \leq t).$$

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in** L_2 to a random variable X if

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^2 = 0.$$

We say that random variables $X_1, X_2, \ldots$ converge **almost surely** (or **with probability one**) to a random variable X if

$$\mathbf{P}(\lim_{n\to\infty} X_n = X) = 1.$$

Let $\lambda > 0$. Let $T_1, T_2, \ldots$ be independent exponential random variables with parameter λ. Let $Y_0 = 0$, and for any $n \geq 1$, let $Y_n := T_1 + \cdots + T_n$. A **Poisson Process** with parameter $\lambda > 0$ is a set of integer-valued random variables $\{N(s)\}_{s\geq 0}$ defined by $N(s) := \max\{n \geq 0 \colon Y_n \leq s\}$, $\forall\, s \geq 0$.

Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbf{E}X_1 = 0$ and $\mathbf{E}\,|X_1| < \infty$. Let $X_0 := 0$ and for any integer $n \geq 0$, define $S_n := X_0 + \cdots + X_n$. We call the sequence of random variables $S_0, S_1, \ldots$ a **random walk** started at 0. More generally, if $c \in \mathbf{R}$ is a constant and if $X_0 = c$, we call the sequence of random variables $S_0, S_1, \ldots$ a **random walk** started at c.

A **stopping time** for a random walk $S_0, S_1, \ldots$ is a random variable T taking values in $0, 1, 2, \ldots, \cup\{\infty\}$ such that, for any integer $n \geq 0$, the event $\{T = n\}$ is determined by $X_0, \ldots, X_n$. More formally, for any integer $n \geq 1$, there is a set $B_n \subseteq \mathbf{R}^{n+1}$ such that $\{T = n\} = \{(X_0, \ldots, X_n) \in B_n\}$. Put another way, the indicator function $1_{\{T=n\}}$ is a function of the random variables $X_0, \ldots, X_n$.

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (3 points) Let $A_1, A_2, \ldots$ be subsets of a sample space Ω. Let $\mathbf{P}$ denote a probability law on Ω. Then

   $$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \mathbf{P}\left(\cup_{n=1}^{\infty} A_n\right)$$

   TRUE FALSE (circle one)

   (b) (3 points) Let X be a continuous random variable. Let f_X be the density function of X. Then, for any $t \in \mathbf{R}$, $\frac{d}{dt}\mathbf{P}(X \leq t)$ exists, and

   $$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t).$$

   TRUE FALSE (circle one)

   (c) (3 points) Let X be a random variable such that $\mathbf{E}X^4 < \infty$. Then $\mathbf{E}X^2 < \infty$.

   TRUE FALSE (circle one)

   (d) (3 points) Let X be a random variable such that $\text{var}(X) = 2$ and $\mathbf{E}\,|X| = 4$. Then

   $$\mathbf{P}(|X - \mathbf{E}X| > 2) \leq 1/2.$$

   TRUE FALSE (circle one)

(e) (3 points) Let $i = \sqrt{-1}$. Let $X_1, X_2, \ldots$ be random variables such that, for any $t \in \mathbf{R}$,

$$\lim_{n\to\infty} \mathbf{E}e^{itX_n} = e^{-t^2/2}.$$

Then $X_1, X_2, \ldots$ converges in distribution to a standard Gaussian random variable.

TRUE FALSE (circle one)

(f) (3 points) Let X be a standard Gaussian random variable (so that $\mathbf{P}(X \leq t) = \int_{-\infty}^{t} e^{-x^2/2}dx/\sqrt{2\pi}$ for any $t \in \mathbf{R}$.) Then

$$\mathbf{P}(X > t) < \frac{1}{t}, \quad \forall\, t > 0$$

TRUE FALSE (circle one)

(g) (3 points) Let $\{N(s)\}_{s\geq 0}$ be a Poisson Process with parameter $\lambda = 1$. Then

$$\mathbf{E}\big((N(4) - N(2))N(2)\big) = 4.$$

TRUE FALSE (circle one)

(h) (4 points) If a sequence of random variables $X_1, X_2, \ldots$ converges in distribution to a random variable X, then $X_1, X_2, \ldots$ converges almost surely to X.

TRUE FALSE (circle one)

(i) (4 points) If a sequence of random variables $X_1, X_2, \ldots$ converges in distribution to a random variable X, then

$$\lim_{n\to\infty} \mathbf{E}X_n = \mathbf{E}X$$

TRUE FALSE (circle one)

2. (10 points) Let A, B be events in a sample space. Let $C_1, \ldots, C_n$ be events such that $C_i \cap C_j = \emptyset$ for any $i, j \in \{1, \ldots, n\}$, and such that $\cup_{i=1}^{n} C_i = B$. Show:

$$\mathbf{P}(A|B) = \sum_{i=1}^{n} \mathbf{P}(A|B \cap C_i)\mathbf{P}(C_i|B).$$

3. (10 points) Let X be a random variable that is uniformly distributed in $[0, 1]$. Let $Y := 4X(1 - X)$. Find f_Y, the density function of Y.

4. (10 points) Let X, Y be independent random variables. Suppose X has moment generating function

$$M_X(t) = 1 + t^6, \qquad \forall\, t \in \mathbf{R}.$$

Suppose Y has moment generating function

$$M_Y(t) = 1 + t^2, \qquad \forall\, t \in \mathbf{R}.$$

Compute $\mathbf{E}\Big[(X+Y)^2\Big]$.

5. (10 points) Using the Central Limit Theorem, prove the Weak Law of Large Numbers.

   (You may assume that $X_1, X_2, \ldots$ are independent, identically distributed random variables such that $\mathbf{E}\,|X_1| < \infty$ and $0 < \text{var}(X_1) < \infty$.)

6. (10 points) Suppose you flip a fair coin 120 times. During each coin flip, this coin has probability 1/2 of landing heads, and probability 1/2 of landing tails.

   Let A be the event that you get more than 90 heads in total. Show that

$$\mathbf{P}(A) \leq \frac{1}{60}.$$

7. (10 points) Let $X_1, X_2, \ldots$ be a Bernoulli process with parameter $p = 1/2$. What is the expected number of trials that have to occur before we see two consecutive "successes"? (The number of trials that you are counting should include the second "success" that occurs.)

   (Your final answer can be left as an infinite sum of numbers. You get three bonus points if your final answer is a single real number that is justified correctly.)

8. (10 points) Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables such that, for any $i \geq 1$

$$\mathbf{P}(X_i \leq t) = \begin{cases} \frac{t}{t+1} & , \text{if } t \geq 0. \\ 0 & , \text{if } t < 0. \end{cases}$$

For any $n \geq 1$, let $M_n := \max(X_1, \ldots, X_n)$.

(i) Explicitly compute $\mathbf{P}(M_n \leq t)$ for any $t \in \mathbf{R}$.

(ii) Show that $\frac{M_n}{n}$ converges in distribution to some random variable W, as $n \to \infty$.

(iii) Explicitly compute $\mathbf{P}(\frac{1}{W} \geq t)$ for any $t \in \mathbf{R}$.

9. (10 points) Let $t_1, t_2, \ldots$ be positive, independent identically distributed random variables. Let $\mu \in \mathbf{R}$. Assume $\mathbf{E}t_1 = \mu$. For any $n \geq 1$, let $T_n := t_1 + \cdots + t_n$. For any positive integer t, let $N_t := \min\{n \geq 1 \colon T_n \geq t\}$.

   Show that N_t/t converges almost surely to $1/\mu$ as $t \to \infty$.

   (Hint: if c, t are positive integers, then $\{N_t \leq ct\} = \{T_{ct} \geq t\}$. Apply the Strong Law to T_{ct}.)

10. (10 points) Let $X_0 := x_0 \in \mathbf{Z}$. Let $X_1, X_2, \ldots$ be independent random variables such that $\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2$ for all $n \geq 1$. Let $S_0, S_1, \ldots$ be the corresponding random walk started at x_0. Let $a, b \in \mathbf{Z}$ such that $a < x_0 < b$.

Let $T := \min\{n \geq 1 \colon S_n \in \{a, b\}\}$. Show:

$$\mathbf{P}(S_T = a) = \frac{x_0 - b}{a - b}.$$

(You may assume that $\mathbf{P}(T < \infty) = 1$.)

(Scratch paper)

(More scratch paper)