Please provide complete and well-written solutions to the following exercises.

Due November 2, in the discussion section.

Homework 4

Exercise 1. Compute the characteristic function of a uniformly distributed random variable on $[-1,1]$. (Some of the following formulas might help to simplify your answer: $e^{it} = \cos(t) + i\sin(t)$, $\cos(t) = [e^{it} + e^{-it}]/2$, $\sin(t) = [e^{it} - e^{-it}]/[2i]$, $t \in \mathbf{R}$.) (Here $i := \sqrt{-1}$.)

Exercise 2. Let X be a random variable. Assume we can differentiate under the expected value of $\mathbf{E}e^{itX}$ any number of times. For any positive integer n, show that

$$\frac{d^n}{dt^n}|_{t=0}\phi_X(t) = i^n\mathbf{E}(X^n).$$

So, in principle, all moments of X can be computed just by taking derivatives of the characteristic function.

Exercise 3. Let X be a random variable such that $\mathbf{E}\,|X|^3 < \infty$. Prove that for any $t \in \mathbf{R}$,

$$\mathbf{E}e^{itX} = 1 + it\mathbf{E}X - t^2\mathbf{E}X^2/2 + o(t^2).$$

That is,

$$\lim_{t\to 0} t^{-2}\left|\mathbf{E}e^{itX} - [1 + it\mathbf{E}X - t^2\mathbf{E}X^2/2]\right| = 0$$

(Hint: it may be helpful to use Jensen's inequality to first justify that $\mathbf{E}\,|X| < \infty$ and $\mathbf{E}X^2 < \infty$. Then, use the Taylor expansion with error bound: $e^{iy} = 1 + iy - y^2/2 - (i/2)\int_0^y (y-s)^2 e^{is} ds$, which is valid for any $y \in \mathbf{R}$.)

Actually, this same bound holds only assuming $\mathbf{E}X^2 < \infty$, but the proof of that bound requires things we have not discussed.

Exercise 4 (Convolution is Associative)**.** Let $g, h, d \colon \mathbf{R} \to \mathbf{R}$. Then for any $t \in \mathbf{R}$,

$$((g * h) * d)(t) = (g * (h * d))(t)$$

Exercise 5. Let X, Y, Z be independent and uniformly distributed on $[0,1]$. Note that f_X is not a continuous function.

Using convolution, compute f_{X+Y}. Draw f_{X+Y}. Note that f_{X+Y} is a continuous function, but it is not differentiable at some points.

Using convolution, compute f_{X+Y+Z}. Draw f_{X+Y+Z}. Note that f_{X+Y+Z} is a differentiable function, but it does not have a second derivative at some points.

Make a conjecture about how many derivatives $f_{X_1+\cdots+X_n}$ has, where $X_1, \ldots, X_n$ are independent and uniformly distributed on $[0,1]$. You do not have to prove this conjecture. The idea of this exercise is that convolution is a kind of average of functions. And the more averaging you do, the more derivatives $f_{X_1+\cdots+X_n}$ has.

Exercise 6. Construct two random variables X, Y such that X and Y are each uniformly distributed on $[0, 1]$, and such that $\mathbf{P}(X + Y = 1) = 1$.

Then construct two random variables W, Z such that W and Z are each uniformly distributed on $[0, 1]$, and such that $W + Z$ is uniformly distributed on $[0, 2]$.

(Hint: there is a way to do each of the above problems with about one line of work. That is, there is a way to solve each problem without working very hard.)