Probability, 170B, Fall 2017, UCLA — Instructor: Steven Heilman

Name: ____________________ UCLA ID: ______________ Date: ____________

Signature: ____________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	9	
2	10	
3	10	
4	10	
5	10	
Total:	49	

Do not write in the table to the right. Good luck![a]

Reference sheet

Below are some definitions that may be relevant.

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in probability** to a random variable X if: for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) = 0.$$

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in distribution** to a random variable X if, for any $t \in \mathbf{R}$ such that $\mathbf{P}(X \leq t)$ is continuous at t,

$$\lim_{n\to\infty} \mathbf{P}(X_n \leq t) = \mathbf{P}(X \leq t).$$

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in L_2** to a random variable X if

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^2 = 0.$$

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (3 points) Let X be a random variable. Let $i := \sqrt{-1}$. Then $\left|\mathbf{E}e^{itX}\right| \leq 1$ for any $t \in \mathbf{R}$.

   TRUE FALSE (circle one)

   (b) (3 points) Suppose I am flipping a coin over and over again. For any positive integer n, let A_n be the event that the n^{th} coin flip is heads. Suppose $\mathbf{P}(A_n) = n^{-2}$, for any positive integer n. Let B be the event that infinitely many of the coin flips are heads. Then $\mathbf{P}(B) = 0$.

   TRUE FALSE (circle one)

   (c) (3 points) Let $X_1, X_2, \ldots$ be independent random variables. Let $\mu := \mathbf{E}X_1$ and let $\sigma := \mathrm{var}(X_1)$. Assume $0 < \sigma < \infty$ and $-\infty < \mu < \infty$. Then, for any $t \in \mathbf{R}$,

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq t\right) = \int_{-\infty}^{t} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}.$$

   TRUE FALSE (circle one)

2. (10 points) Let $X_1, X_2, \ldots$ be independent, identically distributed random variables such that $\mathbf{E}|X_1| < \infty$ and $\mathrm{var}(X_1) < \infty$. For any $n \geq 1$, define

$$Y_n := \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

Show that $Y_1, Y_2, \ldots$ converges in probability. Express the limit in terms of $\mathbf{E}X_1$ and $\mathrm{var}(X_1)$.

3. (10 points) Let X be a random variable. Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^4 = 0.$$

Prove that $X_1, X_2, \ldots$ converges in probability to X.

4. (10 points) Let $f, g\colon \mathbf{R} \to \mathbf{R}$. Recall that $(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$. Show that, for any $t \in \mathbf{R}$,

$$(f * g)(t) = (g * f)(t).$$

5. (10 points) Let X, Y be independent random variables. Suppose X is uniformly distributed in $[0,1]$. And suppose Y has density given by

$$f_Y(t) = \begin{cases} 0 & \text{, if } t < 0 \\ t & \text{, if } 0 \leq t \leq 1 \\ 2-t & \text{, if } 1 \leq t \leq 2 \\ 0 & \text{, if } t > 2. \end{cases}$$

Find the density of $X + Y$.

(Scratch paper)