170B Midterm 2 Solutions[1]

1. Question 1

Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

(a) Let X, Y be two random variables such that $M_X(t) = M_Y(t)$ for all $t \in \mathbf{R}$ (and such that $M_X(t), M_Y(t)$ exist for all $t \in \mathbf{R}$). (Recall that $M_X(t) = \mathbf{E}e^{tX}$ for any $t \in \mathbf{R}$). Then $X = Y$.

FALSE. Let X be a standard Gaussian random variable, and let $Y := -X$. Then $X \neq Y$, but $M_X(t) = e^{t^2/2} = M_Y(t)$ for all $t \in \mathbf{R}$.

(b) Let $f, g \colon \mathbf{R} \to \mathbf{R}$. Recall that $(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$. Then

$$(f * g)(t) = (g * f)(t), \qquad \forall\, t \in \mathbf{R}.$$

TRUE. Changing variables $u = t - x$ so that $du = -dx$,

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx = -\int_{\infty}^{-\infty} f(t-u)g(u)du = \int_{-\infty}^{\infty} g(u)f(t-u)du = (g * f)(t).$$

(c) Let $X_1, X_2, \ldots$ be independent random variables. Let $\mu := \mathbf{E}X_1$. Then, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\left(\left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \varepsilon \right) = 0.$$

FALSE. We made no mention of being identically distributed. To get a counterexample, let $X_1 = 0$ and let $X_n = 1$ for all $n \geq 2$. (Constant functions are automatically independent.) Then $\frac{X_1+\cdots+X_n}{n} - \mu = \frac{n-1}{n}$. So, if $\varepsilon = 1/2$ and $n > 3$, $\mathbf{P}\left(\left| \frac{X_1+\cdots+X_n}{n} - \mu \right| \geq \varepsilon \right) = 1$. That is, $\lim_{n\to\infty} \mathbf{P}\left(\left| \frac{X_1+\cdots+X_n}{n} - \mu \right| \geq \varepsilon \right) = 1$.

2. Question 2

Let X be a random variable such that $\mathbf{E}X = 0$ and $\mathrm{var}(X) = 0$. Show that

$$\mathbf{P}(X = 0) = 1.$$

Solution 1. We argue by contradiction. Suppose there exists $\varepsilon > 0$ such that $\mathbf{P}(|X| > \varepsilon) > 0$. Then

$$\mathrm{var}(X) = \mathbf{E}X^2 \geq \mathbf{E}X^2 1_{\{|X|>\varepsilon\}} \geq \varepsilon^2 \mathbf{E}1_{\{|X|>\varepsilon\}} = \varepsilon^2 \mathbf{P}(|X| > \varepsilon) > 0.$$

Having achieve a contradiction, we conclude that no such $\varepsilon > 0$ exists. That is, $\mathbf{P}(|X| > \varepsilon) = 0$ for every $\varepsilon > 0$. By continuity of the probability law, $\mathbf{P}(|X| > 0) = 0$, so that $\mathbf{P}(X = 0) = 1$.

Solution 2. We argue by contradiction. Using the definition of $\mathbf{E}X^2$,

$$0 = \mathrm{var}(X) = \mathbf{E}X^2 = \int_0^{\infty} \mathbf{P}(X^2 > t)dt.$$

The function of t, $\mathbf{P}(X^2 > t)$, is a decreasing and nonnegative function whose integral is zero. Therefore, $\mathbf{P}(X^2 > t) = 0$ for all $t > 0$. That is, $\mathbf{P}(X^2 = 0) = 1$.

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3. Question 3

Let X be a random variable that is uniformly distributed on $[-1,1]$.
For any $t \in \mathbf{R}$, compute $M_X(t) = \mathbf{E}e^{tX}$.
Then, for any $t \in \mathbf{R}$, compute $\phi_X(t) = \mathbf{E}e^{itX}$, where $i = \sqrt{-1}$.
Solution.

$$M_X(t) = \mathbf{E}e^{tX} = \frac{1}{2}\int_{-1}^{1} e^{tx}dx = [(2t)^{-1}e^{tx}]_{x=-1}^{x=1} = (2t)^{-1}(e^t - e^{-t}) = t^{-1}\sinh(t).$$

$$\phi_X(t) = \mathbf{E}e^{itX} = \frac{1}{2}\int_{-1}^{1} e^{itx}dx = [(2it)^{-1}e^{itx}]_{x=-1}^{x=1} = (2it)^{-1}(e^{it} - e^{-it}) = t^{-1}\sin(t).$$

4. Question 4

Let X, Y be independent exponential random variables with parameter 1. So, X has density

$$f_X(x) := \begin{cases} e^{-x} & , \text{if } x \geq 0 \\ 0 & , \text{if } x < 0. \end{cases}$$

Find the density of $X+Y$.

Solution. From Proposition 2.60, if $t > 0$, then

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(x)f_Y(t-x)dx = \int_0^t e^{-x}e^{-(t-x)}dx = \int_0^t e^{-t}dx = te^{-t}$$

And if $t < 0$, then $f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(x)f_Y(t-x)dx = 0$. In summary,

$$f_{X+Y}(t) = \begin{cases} te^{-t} & , \text{if } t \geq 0 \\ 0 & , \text{if } t < 0. \end{cases}$$

5. Question 5

Suppose you flip a fair coin 80 times. During each coin flip, this coin has probability 1/2 of landing heads, and probability 1/2 of landing tails.

Let A be the event that you get more than 50 heads in total. Show that

$$\mathbf{P}(A) \leq \frac{1}{10}.$$

Solution 1. For any $n \geq 1$, define X_n so that

$$X_n = \begin{cases} 1 & , \text{if the } n^{th} \text{ coin flip is heads} \\ 0 & , \text{if the } n^{th} \text{ coin flip is tails.} \end{cases}$$

By its definition $\mathbf{E}X_n = 1/2$ and $\mathrm{var}(X_n) = (1/2)(1/4) + (1/2)(1/4) = 1/4$.

Let $S := X_1 + \cdots + X_{80}$ be the number of heads that are flipped. Then $\mathbf{E}S = 40$, and $\mathrm{var}(S) = 80\mathrm{var}(X_1) = 20$. Markov's inequality says, for any $t > 0$

$$\mathbf{P}(S > t) \leq \mathbf{E}S/t = 40/t.$$

This is not helpful. Instead, we use Chebyshev's inequality. This says, for any $t > 0$,

$$\mathbf{P}(|S - 40| > t) \leq t^{-2}\mathrm{var}(S) = 20t^{-2}.$$

Choosing $t = 10$ shows that $\mathbf{P}(|S - 40| > 10) \leq 1/5$. Now, using symmetry of S (interchanging the roles of heads and tails),

$$\mathbf{P}(|S - 40| > 10) = \mathbf{P}(S < 30) + \mathbf{P}(S > 50) = 2\mathbf{P}(S > 50).$$

So,

$$2\mathbf{P}(S > 50) = \mathbf{P}(|S - 40| > 10) \leq 1/5.$$

Solution 2. We use the notation of Solution 1, but instead of Chebyshev's inequality, we use the Chernoff bound. Since S is a sum of 80 independent identically distributed random variables, Proposition 2.43 from the notes says

$$M_S(t) = (M_{X_1}(t))^{80}, \qquad \forall\, t \in \mathbf{R}.$$

So, the Chernoff bound says, for any $r, t > 0$,

$$\mathbf{P}(S > r) \leq e^{-tr}(M_{X_1}(t))^{80} = e^{-tr}((1/2)(1 + e^t))^{80} \qquad (*).$$

Setting $f(t) = e^{-rt}(1 + e^t)^{80}$ and solving $f'(t) = 0$ for t shows that $t = \log(5/3)$ minimizes the quantity $f(t)$. So, choosing $r = 50$ and $t = \log(5/3)$ in $(*)$ gives

$$\mathbf{P}(S > 50) \leq e^{-tr}((1/2)(1 + 5/3))^{80} = (5/3)^{-50}(4/3)^{80} \leq 0.08 < 1/10.$$

Solution 3. (The following solution based on the Central Limit Theorem only received partial credit, since it only approximately shows that $\mathbf{P}(A) < 1/10$.) We use the notation of Solution 1, but instead of Chebyshev's inequality, we use the Central Limit Theorem. Since $X_1, X_2, \ldots$ are independent identically distributed random variables with mean $1/2$ and variance $1/4$, the Central Limit Theorem implies that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_1 + \cdots + X_n - n/2}{\sqrt{(1/4)}\sqrt{n}} > t\right) = \int_t^\infty e^{-x^2/2}dx/\sqrt{2\pi}.$$

So, choosing $n = 80$ and $t = \sqrt{5}$, we have the approximation

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_{80} - 40}{\sqrt{(1/4)}\sqrt{80}} > \sqrt{5}\right) \approx \int_{\sqrt{5}}^\infty e^{-x^2/2}dx/\sqrt{2\pi}.$$

Simplifying a bit,

$$\mathbf{P}\left(S - 40 > 10\right) \approx \int_{\sqrt{5}}^\infty e^{-x^2/2}dx/\sqrt{2\pi}.$$

Using $\sqrt{5} > 2$ and the approximation $\int_2^\infty e^{-x^2/2}dx/\sqrt{2\pi} \approx .025$, we have

$$\mathbf{P}(S > 50) \approx \int_{\sqrt{5}}^\infty e^{-x^2/2}dx/\sqrt{2\pi} \leq \int_2^\infty e^{-x^2/2}dx/\sqrt{2\pi} \approx .025 < 1/10.$$