Probability, 170B, Winter 2017, UCLA — Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	9	
2	10	
3	10	
4	10	
5	10	
Total:	49	

Do not write in the table to the right. Good luck![a]

Reference sheet

Below are some definitions that may be relevant.

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in probability** to a random variable X if: for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) = 0.$$

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in distribution** to a random variable X if, for any $t \in \mathbf{R}$ such that the CDF of X is continuous at t,

$$\lim_{n\to\infty} \mathbf{P}(X_n \leq t) = \mathbf{P}(X \leq t).$$

We say that a sequence of random variables $X_1, X_2, \ldots$ **converges in** L_2 to a random variable X if

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^2 = 0.$$

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (3 points) Let X, Y be two random variables such that $M_X(t) = M_Y(t)$ for all $t \in \mathbf{R}$ (and such that $M_X(t), M_Y(t)$ exist for all $t \in \mathbf{R}$). (Recall that $M_X(t) = \mathbf{E}e^{tX}$ for any $t \in \mathbf{R}$). Then $X = Y$.

   TRUE FALSE (circle one)

   (b) (3 points) Let $f, g\colon \mathbf{R} \to \mathbf{R}$. Recall that $(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx \ \forall\ t \in \mathbf{R}$. Then

   $$(f * g)(t) = (g * f)(t), \qquad \forall t \in \mathbf{R}.$$

   TRUE FALSE (circle one)

   (c) (3 points) Let $X_1, X_2, \ldots$ be independent random variables. Let $\mu := \mathbf{E}X_1$. Then, for any $\varepsilon > 0$,

   $$\lim_{n\to\infty} \mathbf{P}\left(\left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \varepsilon \right) = 0.$$

   TRUE FALSE (circle one)

2. (10 points) Let X be a random variable such that $\mathbf{E}X = 0$ and $\mathrm{var}(X) = 0$. Show that

$$\mathbf{P}(X = 0) = 1.$$

3. (10 points) Let X be a random variable that is uniformly distributed on $[-1, 1]$.
   For any $t \in \mathbf{R}$, compute $M_X(t) = \mathbf{E}e^{tX}$.
   Then, for any $t \in \mathbf{R}$, compute $\phi_X(t) = \mathbf{E}e^{itX}$, where $i = \sqrt{-1}$.

4. (10 points) Let X, Y be independent exponential random variables with parameter 1. So, X has density

$$f_X(x) := \begin{cases} e^{-x} & , \text{if } x \geq 0 \\ 0 & , \text{if } x < 0. \end{cases}$$

Find the density of $X + Y$.

5. (10 points) Suppose you flip a fair coin 80 times. During each coin flip, this coin has probability $1/2$ of landing heads, and probability $1/2$ of landing tails.

   Let A be the event that you get more than 50 heads in total. Show that

$$\mathbf{P}(A) \leq \frac{1}{10}.$$

(Scratch paper)