170B Midterm 1 Solutions[1]

1. Question 1

Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

(a) Let X, Y be random variables. Then $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$.

FALSE. Let X be any random variable with nonzero variance. Let $Y := X$. Then $\mathrm{var}(X+Y) = \mathrm{var}(2X) = 4\mathrm{var}(X) \neq 2\mathrm{var}(X) = \mathrm{var}(X) + \mathrm{var}(Y)$.

(b) Let X be a continuous random variable. Let f_X be the density function of X. Then, for any $t \in \mathbf{R}$, $\frac{d}{dt}\mathbf{P}(X \leq t)$ exists, and

$$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t).$$

FALSE. Let $f_X(t) := 1$ for any $t \in [0,1]$ and let $f_X(t) := 0$ otherwise. Then

$$\mathbf{P}(X \leq t) = \begin{cases} 0 & , \text{if}\, t < 0 \\ t & , \text{if}\, 0 \leq t \leq 1 \\ 1 & , \text{if}\, t > 1 \end{cases}.$$

In particular, $\frac{d}{dt}\mathbf{P}(X < t)$ does not exist at $t = 0$.

(c) Let X be a random variable such that $\mathbf{E}X^4 < \infty$. Then $\mathbf{E}X^2 < \infty$.

TRUE. By Jensen's inequality (from Exercise 8 on HW2), $(\mathbf{E}X^2)^2 \leq \mathbf{E}X^4 < \infty$.

(d) There exist random variables X, Y such that $\mathrm{var}(X) = 1$, $\mathrm{var}(Y) = 1$ and $\mathrm{cov}(X,Y) = 2$. (Recal that $\mathrm{cov}(X,Y) = \mathbf{E}((X - \mathbf{E}X)(Y - \mathbf{E}Y))$.)

FALSE. By Exercise 1 on HW2, $2 = |\mathrm{cov}(X,Y)| \leq (\mathrm{var}(X))^{1/2}(\mathrm{var}(Y))^{1/2} = 1$, a contradiction.

2. Question 2

Let $\mathbf{P}$ be a probability law on a sample space Ω. Let $A_1, A_2, \ldots$ be sets in Ω that are increasing, so that $A_1 \subseteq A_2 \subseteq \cdots$. You can use as given the following fact:

$$\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(\cup_{n=1}^{\infty} A_n).$$

Using this given fact, prove the following. Let $A_1, A_2, \ldots$ be sets in Ω that are decreasing, so that $A_1 \supseteq A_2 \supseteq \cdots$. Then

$$\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(\cap_{n=1}^{\infty} A_n).$$

Solution. Since $A_1 \supseteq A_2 \supseteq \cdots$, by taking complements we have $A_1^c \supseteq A_2^c \supseteq \cdots$. So, we can apply the given fact to $A_1^c, A_2^c, \ldots$ to get

$$\lim_{n\to\infty} \mathbf{P}(A_n) = 1 - \lim_{n\to\infty} \mathbf{P}(A_n^c) = 1 - \mathbf{P}(\cup_{n=1}^{\infty} A_n^c) = 1 - \mathbf{P}((\cap_{n=1}^{\infty} A_n)^c) = \mathbf{P}(\cap_{n=1}^{\infty} A_n).$$

In the third equality, we applied De Morgan's Law.

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3. Question 3

Let X be a random variable that is uniformly distributed in $[0,1]$. Let $Y := 4X(1-X)$. Find f_Y, the density function of Y.

Solution. Using the quadratic formula, the function $f(t) = 4t(1-t)$ takes the value $c \in [0,1]$ when $x = (1/2) \pm (1/2)\sqrt{1-c}$. So, if $x \in [0,1]$, we have

$$\begin{aligned}\mathbf{P}(4X(1-X) \leq x) &= \mathbf{P}(X \in [0, 1/2 - (1/2)\sqrt{1-x}] \text{ or } X \in [1/2 + (1/2)\sqrt{1-x}, 1]) \\ &= (1/2) - (1/2)\sqrt{1-x} + 1 - (1/2 + (1/2)\sqrt{1-x}) = 1 - \sqrt{1-x}.\end{aligned}$$

We differentiate the CDF to find the density. Then if $0 \leq x \leq 1$, we have

$$f_Y(x) = \frac{d}{dx}(1 - \sqrt{1-x}) = \frac{1}{2}(1-x)^{-1/2}.$$

And $f_Y(x) = 0$ for any other x.

4. Question 4

Let X, Y be independent random variables. Suppose X has moment generating function

$$M_X(t) = 1 + t^6, \qquad \forall\, t \in \mathbf{R}.$$

Suppose Y has moment generating function

$$M_Y(t) = 1 + t^2, \qquad \forall\, t \in \mathbf{R}.$$

Compute $\mathbf{E}\Big[(X+Y)^2\Big]$.

Solution 1. Since X, Y are independent, we have $M_{X+Y}(t) = M_X(t)M_Y(t) = (1+t^6)(1+t^2) = 1 + t^2 + t^6 + t^8$ for all $t \in \mathbf{R}$ by Proposition 2.43 in the notes. As mentioned in the notes,

$$\mathbf{E}(X+Y)^2 = \frac{d^2}{dt^2}|_{t=0} M_{X+Y}(t).$$

Therefore,

$$\mathbf{E}(X+Y)^2 = \frac{d^2}{dt^2}|_{t=0}\Big(1 + t^2 + t^6 + t^8\Big) = [2 + 30t^4 + 56t^6]_{t=0} = 2.$$

Solution 2. As mentioned in the notes,

$$\mathbf{E}X = \frac{d}{dt}|_{t=0} M_X(t) = \frac{d}{dt}|_{t=0}(1+t^6) = [6t^5]_{t=0} = 0.$$

$$\mathbf{E}X^2 = \frac{d^2}{dt^2}|_{t=0} M_X(t) = \frac{d^2}{dt^2}|_{t=0}(1+t^6) = [30t^4]_{t=0} = 0.$$

$$\mathbf{E}Y = \frac{d}{dt}|_{t=0} M_Y(t) = \frac{d}{dt}|_{t=0}(1+t^2) = [2t]_{t=0} = 0.$$

$$\mathbf{E}Y^2 = \frac{d^2}{dt^2}|_{t=0} M_Y(t) = \frac{d^2}{dt^2}|_{t=0}(1+t^2) = 2.$$

Therefore, using also that X, Y are independent,

$$\mathbf{E}(X+Y)^2 = \mathbf{E}X^2 + \mathbf{E}Y^2 + 2\mathbf{E}(XY) = 0 + 2 + (\mathbf{E}X)(\mathbf{E}Y) = 2 + 0 \cdot 0 = 2.$$

5. Question 5

You are trapped in a maze. Your starting point is a room with three doors. The first door will lead you to a corridor which lets you exit the maze after three hours of walking. The second door leads you through a corridor which puts you back to the starting point of the maze after seven hours of walking. The third door leads you through a corridor which puts you back to the starting point of the maze after nine hours of walking. Each time you are at the starting point, you choose one of the three doors with equal probability.

Let X be the number of hours it takes for you to exit the maze. Let Y be the number of the door that you initially choose.

- Compute $\mathbf{E}(X|Y=i)$ for each $i \in \{1,2,3\}$, in terms of $\mathbf{E}X$.
- Compute $\mathbf{E}X$.

(In this problem, you can use any version of the Total Expectation Theorem.)

Solution. $\mathbf{E}(X|Y=1)=3$, $\mathbf{E}(X|Y=2)=7+\mathbf{E}X$, and $\mathbf{E}(X|Y=3)=9+\mathbf{E}X$. To see the first equality, note that if $Y=1$, then you exit the maze in three hours, so $p_{X|Y}(x|1)=1$ when $x=3$, so $\mathbf{E}(X|Y=1)=3$. For the second equality, if $Y=2$, then $p_{X|Y}(x|2)=p_X(x+7)$ for all real x. Finally, from the Total Expectation Theorem, $\mathbf{E}X=\sum_{i=1}^{3}p_Y(i)\mathbf{E}(X|Y=i)=(1/3)(3)+(1/3)(7+\mathbf{E}X)+(1/3)(9+\mathbf{E}X)$. That is, $\mathbf{E}X=19/3+(2/3)\mathbf{E}X$. So, $\mathbf{E}X=19$.