Probability, 170B, Fall 2017, UCLA — Instructor: Steven Heilman

Name: ______________ UCLA ID: ______________ Date: ______________

Signature: ______________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 1

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	8	
2	10	
3	10	
4	10	
5	10	
Total:	48	

Do not write in the table to the right. Good luck![a]

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**. (Unlike the other parts of this exam, on this first question of the exam, you are allowed to use results from the homework.)

   (a) (2 points) Let X, Y be random variables. Then $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$.

   TRUE FALSE (circle one)

   (b) (2 points) Let X be a continuous random variable. Let f_X be the density function of X. Then, for any $t \in \mathbf{R}$, $\frac{d}{dt}\mathbf{P}(X \leq t)$ exists, and

   $$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t).$$

   TRUE FALSE (circle one)

   (c) (2 points) Let X be a random variable such that $\mathbf{E}X^4 < \infty$. Then $\mathbf{E}X^2 < \infty$.

   TRUE FALSE (circle one)

   (d) (2 points) There exist random variables X, Y such that $\mathrm{var}(X) = 1$, $\mathrm{var}(Y) = 1$ and $\mathrm{cov}(X, Y) = 2$. (Recal that $\mathrm{cov}(X, Y) = \mathbf{E}((X - \mathbf{E}X)(Y - \mathbf{E}Y))$.)

   TRUE FALSE (circle one)

2. (10 points) Let $\mathbf{P}$ be a probability law on a sample space Ω. Let $A_1, A_2, \ldots$ be sets in Ω that are increasing, so that $A_1 \subseteq A_2 \subseteq \cdots$. You can use as given the following fact:

$$\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(\cup_{n=1}^{\infty} A_n).$$

Using this given fact, prove the following. Let $A_1, A_2, \ldots$ be sets in Ω that are decreasing, so that $A_1 \supseteq A_2 \supseteq \cdots$. Then

$$\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(\cap_{n=1}^{\infty} A_n).$$

3. (10 points) Let X be a random variable that is uniformly distributed in $[0,1]$. Let $Y := 4X(1-X)$. Find f_Y, the density function of Y.

4. (10 points) Let X, Y be independent random variables. Suppose X has moment generating function

$$M_X(t) = 1 + t^6, \qquad \forall\, t \in \mathbf{R}.$$

Suppose Y has moment generating function

$$M_Y(t) = 1 + t^2, \qquad \forall\, t \in \mathbf{R}.$$

Compute $\mathbf{E}\Big[(X+Y)^2\Big]$.

5. (10 points) You are trapped in a maze. Your starting point is a room with three doors. The first door will lead you to a corridor which lets you exit the maze after three hours of walking. The second door leads you through a corridor which puts you back to the starting point of the maze after seven hours of walking. The third door leads you through a corridor which puts you back to the starting point of the maze after nine hours of walking. Each time you are at the starting point, you choose one of the three doors with equal probability.

   Let X be the number of hours it takes for you to exit the maze. Let Y be the number of the door that you initially choose.

   - Compute $\mathbf{E}(X|Y = i)$ for each $i \in \{1, 2, 3\}$, in terms of $\mathbf{E}X$.
   - Compute $\mathbf{E}X$.

   (In this problem, you can use any version of the Total Expectation Theorem.)

(Scratch paper)