170B Midterm 1 Solutions[1]

1. Question 1

Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

(a) Let $A_1, A_2, \ldots$ be subsets of a sample space Ω. Let $\mathbf{P}$ denote a probability law on Ω. Then

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \mathbf{P}\left(\cup_{n=1}^{\infty} A_n\right)$$

FALSE. Let $A_1 = A_2 = \Omega$ and let $\emptyset = A_3 = A_4 = \cdots$. Then the left side is $1 + 1 = 2$, but the right side is $\mathbf{P}(\Omega) = 1$.

(b) Let X be a continuous random variable. Let f_X be the density function of X. Then, for any $t \in \mathbf{R}$, $\frac{d}{dt}\mathbf{P}(X \leq t)$ exists, and

$$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t).$$

FALSE. Let $f_X(t) := 1$ for any $t \in [0,1]$ and let $f_X(t) := 0$ otherwise. Then

$$\mathbf{P}(X \leq t) = \begin{cases} 0 & , \text{if } t < 0 \\ t & , \text{if } 0 \leq t \leq 1 \\ 1 & , \text{if } t > 1 \end{cases}.$$

In particular, $\frac{d}{dt}\mathbf{P}(X \leq t)$ does not exist at $t = 0$.

(c) Let $A_1, A_2, \ldots$ be subsets of a sample space Ω. Let $\mathbf{P}$ denote a probability law on Ω. Then

$$\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}\left(\cup_{n=1}^{\infty} A_n\right)$$

FALSE. Let $A_1 = \Omega$ and let $\emptyset = A_2 = A_3 = \cdots$. Then the left side is $\lim_{n\to\infty} \mathbf{P}(\emptyset) = 0$, but the right side is $\mathbf{P}(\Omega) = 1$.

(d) $[0,1] = \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$.

TRUE. If $0 \leq x \leq 1$, then $x \in \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$ for any positive integer j. So $x \in \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$, by the definition of countable intersection. That is, $[0,1] \subseteq \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$. On the other hand, if $x \in \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$, then $-1/j \leq x \leq 1 + 1/j$ for every positive integer j. The only $x \in \mathbf{R}$ satisfying this condition are $x \in [0,1]$. (By the Archimedean property of the real numbers, if $x < 0$ then there exists a positive integer j such that $x < -1/j < 0$; and if $x > 1$, then there exists a positive integer j such that $x > 1 + 1/j$.) So, $\bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right) \subseteq [0,1]$. In conclusion, $[0,1] = \bigcap_{j=1}^{\infty} \left(-\frac{1}{j}, 1 + \frac{1}{j}\right)$.

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2. Question 2

Let X and Y be nonnegative random variables. Recall that we can define

$$\mathbf{E}X := \int_0^\infty \mathbf{P}(X > t)dt.$$

Assume that $X \leq Y$. Conclude that $\mathbf{E}X \leq \mathbf{E}Y$.

Solution 1. For any $t \geq 0$, we have the containment $\{X > t\} \subseteq \{Y > t\}$. (If $\omega \in \Omega$ satisfies $X(\omega) > t$, then since $Y \geq X$, we have $Y(\omega) \geq X(\omega) > t$, so that $Y(\omega) > t$.) Since $\{X > t\} \subseteq \{Y > t\}$, we have $\mathbf{P}(X > t) \leq \mathbf{P}(Y > t)$. Integrating this inequality over all $t \geq 0$, we get

$$\mathbf{E}X = \int_0^\infty \mathbf{P}(X > t)dt \leq \int_0^\infty \mathbf{P}(Y > t)dt = \mathbf{E}Y.$$

Solution 2. Since $Y - X \geq 0$, the definition of expected value shows that $\mathbf{E}(Y - X) = \int_0^\infty \mathbf{P}(Y - X > t)dt \geq 0$. So, $\mathbf{E}Y - \mathbf{E}X \geq 0$, as desired.

3. Question 3

Let X be a uniformly distributed random variable on $[-1, 1]$. Let $Y := X^4$. Find f_Y. (Here f_Y denotes the density function of Y.)

Solution. Since $|X| \leq 1$, we have $0 \leq X^4 \leq 1$, so that $0 \leq Y \leq 1$. So, if $t < 0$, $\mathbf{P}(Y \leq t) = 0$. And if $0 \leq t \leq 1$, we have

$$\mathbf{P}(Y \leq t) = \mathbf{P}(X^4 \leq t) = \mathbf{P}(|X| \leq t^{1/4}) = \mathbf{P}(-t^{1/4} \leq X \leq t^{1/4}) = \frac{t^{1/4} - (-t^{1/4})}{2} = t^{1/4}.$$

In the penultimate line, we used that X is uniformly distributed on $[-1, 1]$. In summary,

$$\mathbf{P}(Y \leq t) = \begin{cases} 0 & \text{, if } t < 0 \\ t^{1/4} & \text{, if } 0 \leq t \leq 1 \\ 1 & \text{, if } t > 1. \end{cases}$$

So, $\mathbf{P}(Y \leq t)$ is differentiable if $t \neq 0$ and $t \neq 1$, and for such t,

$$f_Y(t) = \frac{d}{dt}\mathbf{P}(Y \leq t) = \begin{cases} 0 & \text{, if } t < 0 \\ \frac{1}{4}t^{-3/4} & \text{, if } 0 < t < 1 \\ 0 & \text{, if } t > 1. \end{cases}$$

4. Question 4

Let X, Y be independent random variables. Suppose X has moment generating function

$$M_X(t) = 1 + t^4, \qquad \forall\, t \in \mathbf{R}.$$

Suppose Y has moment generating function

$$M_Y(t) = 1 + t^2, \qquad \forall\, t \in \mathbf{R}.$$

Compute $\mathbf{E}\Big[(X + Y)^2\Big]$.

Solution 1. Since X, Y are independent, we have $M_{X+Y}(t) = M_X(t)M_Y(t) = (1+t^4)(1+t^2) = 1+t^2+t^4+t^6$ for all $t \in \mathbf{R}$ by Proposition 2.43 in the notes. As mentioned in the notes,

$$\mathbf{E}(X+Y)^2 = \frac{d^2}{dt^2}|_{t=0}M_{X+Y}(t).$$

Therefore,

$$\mathbf{E}(X+Y)^2 = \frac{d^2}{dt^2}|_{t=0}\Big(1+t^2+t^4+t^6\Big) = [2+12t^2+30t^4]_{t=0} = 2.$$

Solution 2. As mentioned in the notes,

$$\mathbf{E}X = \frac{d}{dt}|_{t=0}M_X(t) = \frac{d}{dt}|_{t=0}(1+t^4) = [4t^3]_{t=0} = 0.$$
$$\mathbf{E}X^2 = \frac{d^2}{dt^2}|_{t=0}M_X(t) = \frac{d^2}{dt^2}|_{t=0}(1+t^4) = [12t^2]_{t=0} = 0.$$
$$\mathbf{E}Y = \frac{d}{dt}|_{t=0}M_Y(t) = \frac{d}{dt}|_{t=0}(1+t^2) = [2t]_{t=0} = 0.$$
$$\mathbf{E}Y^2 = \frac{d^2}{dt^2}|_{t=0}M_Y(t) = \frac{d^2}{dt^2}|_{t=0}(1+t^2) = 2.$$

Therefore, using also that X, Y are independent,

$$\mathbf{E}(X+Y)^2 = \mathbf{E}X^2 + \mathbf{E}Y^2 + 2\mathbf{E}(XY) = 0 + 2 + (\mathbf{E}X)(\mathbf{E}Y) = 2 + 0\cdot 0 = 2.$$

5. Question 5

Let $0 < p < 1$. Suppose you have a biased coin which has a probability p of landings heads, and probability $1-p$ of landing tails, each time it is flipped. Also, suppose you have a fair six-sided die (so each face of the cube has a distinct label from the set $\{1,2,3,4,5,6\}$, and each time you roll the die, any face of the cube is rolled with equal probability.)

Let N be the number of coin flips you need to do until the first head appears. Now, roll the fair die N times. Let S be the sum of the results of the N rolls of the die. Compute $\mathbf{E}S$

Solution. Let $n \geq 1$ be a fixed integer. Conditional on $N = n$, S will be the roll of the die n times, so $\mathbf{E}(S|N=n)$ is n times the expected value of a single die roll. That is, $\mathbf{E}(S|N=n) = \frac{7}{2}n$. Also, note that N has a geometric distribution, essentially by the definition of a geometric distribution. That is, $\mathbf{P}(N=n) = (1-p)^{n-1}p$ for any $n \geq 1$. So, using any form of the Total Expectation Theorem, we have

$$\begin{aligned}\mathbf{E}S &= \sum_{n=1}^{\infty}\mathbf{P}(N=n)\mathbf{E}(S|N=n) = \sum_{n=1}^{\infty}(1-p)^{n-1}p\frac{7}{2}n\\ &= p\frac{7}{2}\sum_{n=1}^{\infty}(1-p)^{n-1}n = -p\frac{7}{2}\frac{d}{dp}\sum_{n=1}^{\infty}(1-p)^n = -p\frac{7}{2}\frac{d}{dp}\frac{1-p}{p}\\ &= p\frac{7}{2}\frac{-p-(1-p)}{p^2} = \frac{7}{2}\frac{1}{p}.\end{aligned}$$

In an equality in the middle, we used the sum of the geometric series $\sum_{n=1}^{\infty}(1-p)^n = \frac{1-p}{1-(1-p)} = \frac{1-p}{p}$.