Please provide complete and well-written solutions to the following exercises.

Due March 10, in the discussion section.

Homework 8

Exercise 1. Let X be a continuous random variable with distribution function $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ $\forall\, x \in \mathbf{R}$. Show that $\mathrm{var}(X) = 1$.

Exercise 2. Let X be a random variable such that $f_X(x) = x$ when $0 \leq x \leq \sqrt{2}$ and $f_X(x) = 0$ otherwise. Compute $\mathbf{E}X^2$ and $\mathbf{E}X^3$.

Exercise 3 (Numerical Integration)**.** In computer graphics in video games, etc., various integrations are performed in order to simulate lighting effects. Here is a way to use random sampling to integrate a function in order to quickly and accurately render lighting effects. Let $\Omega = [0,1]$, and let $\mathbf{P}$ be the uniform probably law on Ω, so that if $0 \leq a < b \leq 1$, we have $\mathbf{P}([a,b]) = b - a$. Let $X_1, \ldots, X_n$ be independent random variables such that $\mathbf{P}(X_i \in [a,b]) = b - a$ for all $0 \leq a < b \leq 1$, for all $i \in \{1, \ldots, n\}$. Let $f\colon [0,1] \to \mathbf{R}$ be a continuous function we would like to integrate. Instead of integrating f directly, we instead compute the quantity

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i).$$

Show that

$$\lim_{n\to\infty} \mathbf{E}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right) = \int_0^1 f(t)dt.$$

$$\lim_{n\to\infty} \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right) = 0.$$

That is, as n becomes large, $\frac{1}{n}\sum_{i=1}^{n} f(X_i)$ is a good estimate for $\int_0^1 f(t)dt$.

Exercise 4. Let X be a random variable such that $X = 1$ with probability 1. Show that X is not a continuous random variable. That is, there does not exist a probability density function f such that $\mathbf{P}(X \leq a) = \int_{-\infty}^{a} f(x)dx$ for all $x \in \mathbf{R}$. (Hint: if X were continuous, then the function $g(t) = \int_{-\infty}^{t} f(x)dx$ would be continuous, by the Fundamental Theorem of Calculus.)

Exercise 5. Verify that a Gaussian random variable X with mean μ and variance σ^2 actually has mean μ and variance σ^2.

Let $a, b \in \mathbf{R}$ with $a \neq 0$. Show that $aX + b$ is a normal random variable with mean $a\mu + b$ and variance $a^2\sigma^2$.

In particular, conclude that $(X - \mu)/\sigma$ is a standard normal.

Exercise 6. Using the De Moivre-Laplace Theorem, estimate the probability that $1,000,000$ coin flips of fair coins will result in more than $501,000$ heads. (Some of the following integrals may be relevant: $\int_{-\infty}^{0} e^{-t^2/2}dt/\sqrt{2\pi} = 1/2$, $\int_{-\infty}^{1} e^{-t^2/2}dt/\sqrt{2\pi} \approx .8413$, $\int_{-\infty}^{2} e^{-t^2/2}dt/\sqrt{2\pi} \approx .9772$, $\int_{-\infty}^{3} e^{-t^2/2}dt/\sqrt{2\pi} \approx .9987$.)

Casinos do these kinds of calculations to make sure they make money and that they do not go bankrupt. Financial institutions and insurance companies do similar calculations for similar reasons.

Exercise 7. Let X, Y be random variables with joint PDF $f_{X,Y}$. Let $a, b \in \mathbf{R}$. Using the definition of expected value, show that $\mathbf{E}(aX + bY) = a\mathbf{E}X + b\mathbf{E}Y$.

Exercise 8. Suppose you go to the bus stop, and the time T between successive arrivals of the bus is anything between 0 and 30 minutes, with all arrival times being equally likely.

Suppose you get to the bus stop, and the bus just leaves as you arrive. How long should you expect to wait for the next bus? What is the probability that you will have to wait at least 15 minutes for the next bus to arrive?

On a different day, suppose you go to the bus stop and someone says the last bus came 10 minutes ago. How long should you expect to wait for the next bus? What is the probability that you will have to wait at least 10 minutes for the next bus to arrive?

Exercise 9. Let $A_1, A_2, \ldots$ be disjoint events such that $\mathbf{P}(A_i) = 2^{-i}$ for each $i \geq 1$. Assume that $\cup_{i=1}^{\infty} A_i = \Omega$. Let X be a random variable such that $\mathbf{E}(X|A_i) = (-1)^{i+1}$ for each $i \geq 1$. Compute $\mathbf{E}X$.

Exercise 10. Let X, Y be random variables. For any $y \geq 0$, assume that $\mathbf{E}(X|Y = y) = e^{-|y|}$. Also, assume that Y has an exponential distribution with parameter $\lambda = 2$. Compute $\mathbf{E}X$.

Exercise 11 (Optional exercise, ungraded)**.** Let X, Y be independent random variables with joint PDF $f_{X,Y}$. Show that

$$\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y).$$

Exercise 12 (Optional exercise, ungraded)**.** Let X and Y be uniformly distributed random variables on $[0, 1]$. Assume that X and Y are independent. Compute the following probabilities:

- $\mathbf{P}(X > 3/4)$
- $\mathbf{P}(Y < X)$
- $\mathbf{P}(X + Y < 1/2)$
- $\mathbf{P}(\max(X, Y) > 1/2)$
- $\mathbf{P}(XY < 1/3)$.

Exercise 13 (Optional exercise, ungraded)**.** Let X_1, Y_1 be random variables with joint PDF f_{X_1,Y_1}. Let X_2, Y_2 be random variables with joint PDF f_{X_2,Y_2}. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ and let $S\colon \mathbf{R}^2 \to \mathbf{R}^2$ so that $ST(x, y) = (x, y)$ and $TS(x, y) = (x, y)$ for every $(x, y) \in \mathbf{R}^2$. Let $J(x, y)$ denote the determinant of the Jacobian of S at (x, y). Assume that $(X_2, Y_2) = T(X_1, Y_1)$. Using the change of variables formula from multivariable calculus, show that

$$f_{X_2,Y_2}(x, y) = f_{X_1,Y_1}(S(x, y))\, |J(x, y)|\,.$$