Please provide complete and well-written solutions to the following exercises.

Due March 3, in the discussion section.

Homework 7

Exercise 1. Let X, Y and Z be independent geometric random variables with the same parameter $0 < p < 1$. Let k, n be nonnegative integers. Compute $\mathbf{P}(X = k | X + Y + Z = n)$.

Exercise 2. Let X, Y, Z be discrete random variables. Let $A \subseteq \Omega$. Let $c \in \mathbf{R}$. Show that conditional expectation is linear; that is, show that $\mathbf{E}(X + Z|A) = \mathbf{E}(X|A) + \mathbf{E}(Z|A)$ and $\mathbf{E}(cX|A) = c\mathbf{E}(X|A)$. (Recall that Expectation itself is linear as well.) (Hint: it may be useful to write $\mathbf{P}(\{Z + X = t\} \cap A) = \sum_{z,x \in \mathbf{R}\colon z+x=t} \mathbf{P}(\{Z = z\} \cap \{X = x\} \cap A)$.)

Now, let $f(y) = \mathbf{E}(X|Y = y)$ for any $y \in \mathbf{R}$. Then $f\colon \mathbf{R} \to \mathbf{R}$ is a function. In more advanced probability classes, we consider the random variable $f(Y)$, which is denoted by $\mathbf{E}(X|Y)$. Show that $\mathbf{E}(X + Z|Y) = \mathbf{E}(X|Y) + \mathbf{E}(Z|Y)$.

Then, show that $\mathbf{E}[\mathbf{E}(X|Y)] = \mathbf{E}(X)$. (That is, you should show that $\mathbf{E}f(Y) = \mathbf{E}(X)$.) So, understanding $\mathbf{E}(X|Y)$ can help us to compute $\mathbf{E}(X)$.

Exercise 3. Give an example of two random variables X, Y that are independent. Prove that these random variables are independent.

Give an example of two random variables X, Y that are not independent. Prove that these random variables are not independent.

Finally, find two random variables X, Y such that $\mathbf{E}(XY) \neq \mathbf{E}(X)\mathbf{E}(Y)$.

Exercise 4. Is it possible to have a random variable X such that X is independent of X? Either find such a random variable X, or prove that it is impossible to find such a random variable X.

Exercise 5. Let $0 < p < 1$. Let n be a positive integer. Let $X_1, \ldots, X_n$ be pairwise independent Bernoulli random variables. Compute the expected value of

$$S_n = \frac{X_1 + \cdots + X_n}{n}.$$

Then, compute the variance of $S_n - \mathbf{E}(S_n)$. Describe in words what this variance computation tells you as $n \to \infty$. Particularly, what does S_n "look like" as $n \to \infty$? (Think about what we found in the Example, "Playing Monopoly Forever". Also, consider the following statistical interpretation. Suppose each X_i is the result of some poll of person i, where $i \in \{1, \ldots, n\}$. Suppose that each person's response is a Bernoulli random variable with parameter p, and each person's response is independent of each other person's response. Then S_n is the average of the results of the poll. If $S_n - \mathbf{E}(S_n)$ has small variance, then our poll is very accurate.

So, how accurate is the poll as $n \to \infty$? Note that the accuracy of the poll does *not* depend on the size of the population you are sampling from!)

Exercise 6. Let X and Y be discrete random variables on a sample space Ω, and let $\mathbf{P}$ be a probability law on Ω. Assume that X and Y are independent. Assume that X and Y take a finite number of values. Let $f, g\colon \mathbf{R} \to \mathbf{R}$ be functions. Show that

$$\mathbf{E}(f(X)g(Y)) = \mathbf{E}(f(X))\mathbf{E}(g(Y)).$$

Exercise 7. Find three random variables X_1, X_2, X_3 such that: X_1 and X_2 are independent; X_1 and X_3 are independent; X_2 and X_3 are independent; but such that X_1, X_2, X_3 are not independent.

Exercise 8. Let $0 < p < 1$. Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with parameter p. Let $S_n = \sum_{i=1}^{n} X_i$. A moment generating function can help us to compute moments in various ways. Fix $t \in \mathbf{R}$ and compute the moment generating function of X_i for each $i \in \{1, \ldots, n\}$. That is, show that

$$\mathbf{E}e^{tX_i} = (1-p) + pe^t.$$

Then, using the product formula for independent random variables, show that

$$\mathbf{E}e^{tS_n} = [(1-p) + pe^t]^n.$$

By differentiating the last equality at $t = 0$, and using the power series expansion of the exponential function, compute $\mathbf{E}S_n$ and $\mathbf{E}S_n^2$.

Exercise 9. $X_1, \ldots, X_n$ be independent discrete random variables. Show that

$$\mathbf{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) = \prod_{i=1}^{n} \mathbf{P}(X_i \leq x_i), \qquad \forall\, x_1, \ldots, x_n \in \mathbf{R}.$$

Exercise 10. Verify that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$. (Hint: let $T = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. It suffices to show that $T^2 = 1$, since $T > 0$.)