Please provide complete and well-written solutions to the following exercises.

Due February 11, in the discussion section.

Homework 5

Exercise 1. Let X be a discrete random variable with finite variance. Let $t \in \mathbf{R}$. Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(t) = \mathbf{E}(X-t)^2 = \mathbf{E}[(X-t)^2]$. Show that the function f takes its minimum value when $t = \mathbf{E}X$. Moreover, f is uniquely minimized when $t = \mathbf{E}X$.

Exercise 2. Let $0 < p < 1$ and let n be a positive integer. Compute the mean of a binomial random variable with parameter p. Then, compute the mean of a Poisson random variable with parameter $\lambda > 0$.

Exercise 3. Let X be a nonnegative random variable on a sample space Ω. Assume that X only takes integer values. Prove that

$$\mathbf{E}(X) = \sum_{n=1}^{\infty} \mathbf{P}(X \geq n).$$

(You can freely use the following: if $a_{ij} \geq 0$ are numbers for each $i,j \geq 0$, then $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij}$.) Using this result, compute the mean of a geometric distribution with parameter $0 < p < 1$.

Exercise 4. The identity $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij}$ was used in the previous exercise when $a_{ij} \geq 0$ for all $i,j \geq 0$. Unfortunately, this identity is not true in general, as we now show.

Find real numbers a_{ij} for all $i,j \geq 0$ such that $\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{ij}\right) \neq \sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty} a_{ij}\right)$.

(Hint: it may be helpful to think of the numbers a_{ij} arranged in an infinite matrix. Also, it suffices to choose $a_{ij} \in \{-1,0,1\}$ for all $i,j \geq 0$. Also, it is possible to choose the numbers a_{ij} such that $\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} a_{ij}\right) = 0$, whereas $\sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty} a_{ij}\right) = 1$.)

Exercise 5. As we will see later in the course, the expectation is very closely related to integrals. This exercise gives a hint toward this relation. Let $\Omega = [0,1]$. Let $\mathbf{P}$ be the probability law on Ω such that $\mathbf{P}([a,b]) = \int_a^b dt = b - a$ whenever $0 \leq a < b \leq 1$. Let n be a positive integer. Let $X\colon \Omega \to \mathbf{R}$ be such that X is constant on any interval of the form $[i/n, (i+1)/n)$, whenever $0 \leq i \leq n-1$. Show that

$$\mathbf{E}(X) = \int_0^1 X(t)dt$$

Now, consider a different probability law, where $\mathbf{P}([a,b]) = \int_a^b \frac{1}{2\sqrt{t}}dt$ whenever $0 \leq a < b \leq 1$. Show that

$$\mathbf{E}(X) = \int_0^1 X(t)\frac{1}{2\sqrt{t}}dt.$$

Exercise 6. Let $b_1, \ldots, b_n$ be distinct numbers, representing the quality of n people. Suppose n people arrive to interview for a job, one at a time, in a random order. That is, every possible arrival order of these people is equally likely. We can think of an arrival ordering of the people as a list of the form $a_1, \ldots, a_n$, where the list $a_1, \ldots, a_n$ is a permutation of the numbers $b_1, \ldots, b_n$. Moreover, we interpret a_1 as the rank of the first person to arrive, a_2 as the rank of the second person to arrive, and so on. And all possible permutations of the numbers $b_1, \ldots, b_n$ are equally likely to occur.

For each $i \in \{1, \ldots, n\}$, upon interviewing the i^{th} person, if $a_i > a_j$ for all $1 \leq j < i$, then the i^{th} person is hired. That is, if the person currently being interviewed is better than the previous candidates, she will be hired. What is the expected number of hirings that will be made? (Hint: let $X_i = 1$ if the i^{th} person to arrive is hired, and let $X_i = 0$ otherwise. Consider $\sum_{i=1}^n X_i$.)

Exercise 7. Let X be a Poisson random variable with parameter $\lambda > 0$. Compute

$$\mathbf{E}(1+X)^{-1} = \mathbf{E}[(1+X)^{-1}].$$