Please provide complete and well-written solutions to the following exercises.

Due February 4, in the discussion section.

Homework 4

Exercise 1. The Wheel of Fortune involves the repeated spinning of a wheel with 72 possible stopping points. We assume that each time the wheel is spun, any stopping point is equally likely. Exactly one stopping point on the wheel rewards a contestant with $\$1,000,000$. Suppose the wheel is spun 24 times. Let X be the number of times that someone wins $\$1,000,000$. Using the Poisson Approximation the Binomial, estimate the following probabilities: $\mathbf{P}(X=0)$, $\mathbf{P}(X=1)$, $\mathbf{P}(X=2)$. (Hint: consider the binomial distribution with $p=1/72$.)

Exercise 2. Count the number of distinct ways in which you can arrange the letters of the words: CATTERPILLAR, and ARUGULA.

Exercise 3. The Fibonacci sequence is a sequence of integers $a_0, a_1, a_2, \ldots$ such that $a_0 = 0$, $a_1 = 1$, and such that, for any $n \geq 2$, we have $a_n = a_{n-1} + a_{n-2}$. Using the linear algebraic technique for solving recursions discussed for the Gambler's Ruin problem, find an explicit expression for a_n for any $n \geq 2$. (Hint: the golden mean $(1+\sqrt{5})/2$ should arise somewhere.)

Exercise 4. Suppose X is a random variable such that: $p_X(-3) = .1$, $p_X(-2) = .2$, $p_X(-1) = .15$, $p_X(0) = .2$, $p_X(3) = .1$, $p_X(5) = .15$, $p_X(6) = .05$, and $p_X(10) = .05$.

Compute the probabilities of the following events

- $X > 3$
- $4 < X < 7$ or $X > 9$
- $0 < X < 4$ or $7 < X \leq 10$.

Exercise 5. Suppose the probability of that you receive a prize in the mail is $1/7000000$. Show that you need to receive about 7000000 pieces of mail so that the probability that you receive at least one prize in the mail is about $1 - 1/e$. (Let A_i be the event that you receive a prize in the mail in the i^{th} piece of mail. You should assume that the events $A_1, A_2, \ldots$ are all independent.)

Exercise 6. Let $\Omega = \{-3,-2,-1,0,1,2,3\}$. Suppose $X(\omega) = \omega$ for all $\omega \in \Omega$. Let $f\colon \mathbf{R} \to \mathbf{R}$ so that $f(x) = x^2$ for any $x \in \mathbf{R}$. Compute the PMF of $f(X)$.

Exercise 7. Let X have a geometric distribution with parameter p. Let n be a positive integer. Let $Y = X^4$ and let $Z = \min(X, n)$.

Find the PMF of Y. Then, find the PMF of Z.