Please provide complete and well-written solutions to the following exercises.

Due January 28, in the discussion section.

Homework 3

Exercise 1. Let $\Omega = [0,1]\times[0,1]$ so that $\Omega \subseteq \mathbf{R}^2$. Define a probability law $\mathbf{P}$ so that, for any set $A \subseteq \Omega$, $\mathbf{P}(A)$ is defined to be the area of A. Let $0 \leq a_1 \leq a_2 \leq 1$ and let $0 \leq b_1 \leq b_2 \leq 1$. Consider the rectangles $A = \{(x,y) \in \Omega \colon a_1 \leq x \leq a_2\}$, $B = \{(x,y) \in \Omega \colon b_1 \leq y \leq b_2\}$. Show that the rectangles A, B are independent.

Exercise 2. Let $\Omega = \mathbf{R}^2$ so that $\Omega \subseteq \mathbf{R}^2$. Define a probability law $\mathbf{P}$ so that, for any set $A \subseteq \Omega$,

$$\mathbf{P}(A) = \frac{1}{2\pi}\iint_A e^{-(x^2+y^2)/2}dxdy.$$

We previous verified that $\mathbf{P}(\Omega) = 1$. Let $-\infty < a_1 \leq a_2 < \infty$ and let $-\infty < b_1 \leq b_2 < \infty$. Consider the infinite rectangles $A = \{(x,y) \in \Omega \colon a_1 \leq x \leq a_2\}$, $B = \{(x,y) \in \Omega \colon b_1 \leq y \leq b_2\}$. Show that the rectangles A, B are independent.

Exercise 3. Let Ω be a sample space and let $\mathbf{P}$ be a probability law on Ω. Let $A, B \subseteq \Omega$. Assume that $A \subseteq B$. Is it possible that A is independent of B? Justify your answer.

Exercise 4 (Inclusion-Exclusion Formula)**.** In the Properties for Probability laws, we showed that $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. The following equality is a generalization of this fact. Let Ω be a discrete sample space, and let $\mathbf{P}$ be a probability law on Ω. Prove the following. Let $A_1, \ldots, A_n \subseteq \Omega$. Then:

$$\mathbf{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mathbf{P}(A_i) - \sum_{1\leq i<j\leq n} \mathbf{P}(A_i \cap A_j) + \sum_{1\leq i<j<k\leq n} \mathbf{P}(A_i \cap A_j \cap A_k)$$
$$\cdots + (-1)^{n+1}\mathbf{P}(A_1 \cap \cdots \cap A_n).$$

(Hint: begin with the identity $0 = (1-1)^m = \sum_{k=0}^m (-1)^k \binom{m}{k}$, which follows from the Binomial Theorem. That is, $1 = \sum_{k=1}^m (-1)^{k+1}\binom{m}{k}$. Now, let $x \in \Omega$ such that x is in exactly m of the sets $A_1, \ldots, A_n$. Compute the "number of times" that the element $x \in \Omega$ is counted for both sides of the Inclusion-Exclusion Formula.)

Exercise 5 (Derangements)**.**

- Suppose you have a car with four tires, and the car mechanic removes all four tires. Suppose the mechanic now puts the tires back on randomly, so that all arrangements of the tires are equally likely. With what probability will no tire end up in its original position? (Hint: let A_i be the event that the i^{th} tire is in the correct position, where $i = 1, 2, 3, 4$. Then, use the Inclusion-Exclusion formula along with De Morgan's Law.)

- Let n be a positive integer. Suppose your car has n tires that are removed. Suppose the mechanic now puts the tires back on randomly, so that all arrangements of the tires are equally likely. With what probability will no tire end up in its original position?
- Compute the latter probability as $n \to \infty$.

Exercise 6. Let A, B, C be events that are pairwise independent, so that $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$, $\mathbf{P}(A \cap C) = \mathbf{P}(A)\mathbf{P}(C)$, $\mathbf{P}(C \cap B) = \mathbf{P}(C)\mathbf{P}(B)$. Assume that

$$\mathbf{P}(A) = 1/2, \quad \mathbf{P}(B) = 1/3, \quad \mathbf{P}(C) = 1/4, \quad \mathbf{P}(A \cup B \cup C) = 35/48.$$

Are the sets A, B, C independent? Explain.