170A Final Solutions[1]

1. Question 1

True/False

(a) Let X be a continuous random variable with PDF f_X. Then, for any $x \in \mathbf{R}$, $\frac{d}{dx}\mathbf{P}(X \leq x)$ exists, and

$$\frac{d}{dx}\mathbf{P}(X \leq x) = f_X(x)$$

FALSE. Let X be uniformly distributed in $[0,1]$. Then $\mathbf{P}(X \leq x) = 0$ when $x < 0$, and $\mathbf{P}(X \leq x) = x$ when $0 \leq x \leq 1$. So, $(d/dx)\mathbf{P}(X \leq x)$ does not exist when $x = 0$.

(b) Let X be a random variable such that

$$\mathbf{P}(X \leq x) = \begin{cases} 0 & ,\text{if } x < 0 \\ x^3 & ,\text{if } 0 \leq x \leq 1. \\ 1 & ,\text{if } x \geq 1 \end{cases}$$

Then $\mathbf{E}X = \frac{1}{15}$.

FALSE. We have $f_X(x) = (d/dx)\mathbf{P}(X \leq x) = 3x^2$ for any $0 \leq x \leq 1$, and $f_X(x) = 0$ otherwise, so $\mathbf{E}X = \int_0^1 x3x^2 dx = 3/4$.

Let X, Y and Z be random variables. Suppose these random variables have joint density function

$$f_{X,Y,Z}(x,y,z) = \begin{cases} y + z & ,\text{if } 0 \leq x,y,z \leq 1, \\ 0 & ,\text{otherwise.} \end{cases}$$

Then $\mathbf{E}X = \frac{1}{2}$.

TRUE. $\mathbf{E}X = \int_0^1 \int_0^1 \int_0^1 x(y+z)dxdydz = \frac{1}{2}\int_0^1 \int_0^1 (y+z)dydz = 1/2$.

(d) Let X and Y be random variables on a sample space Ω. Let $\mathbf{P}$ be a probability law on Ω. Assume that X and Y are independent (with respect to the probability law $\mathbf{P}$). Let $\mathbf{P}'$ be another (possibly different) probability law on Ω. Then X and Y are independent, with respect to $\mathbf{P}'$.

(We say X and Y are independent with respect to $\mathbf{P}$ if we use $\mathbf{P}$ in the definition of independence of X and Y.)

FALSE. Let $\Omega = \{0,1\}^2$, let $\mathbf{P}$ be uniform on Ω. Let $X(\omega_1, \omega_2) = \omega_1$ and let $Y(\omega_1, \omega_2) = \omega_2$. Then X and Y are independent (as shown in class). But if $\mathbf{P}'$ is defined by $\mathbf{P}'(\{0,0\}) = \mathbf{P}'(\{0,1\}) = \mathbf{P}'(\{1,0\}) = 1/3$ with $\mathbf{P}'(\{1,1\}) = 0$, then $\mathbf{P}'(X = 0, Y = 0) = 1/3 \neq (2/3)(2/3) = \mathbf{P}'(X = 0)\mathbf{P}'(Y = 0)$.

2. Question 2

Two people are flipping fair coins. Let n be a positive integer. Person I flips $n+1$ coins. Person II flips n coins. Show that the following event has probability 1/2: Person I has more heads than Person II.

Solution. Let A be the event that Person I has more heads than Person II. Let S_I be the number of heads from the first n coin flips of person I. Let S_{II} be the number of heads from the first n coin flips of person II. Let B_1 be the event that the $(n+1)^{st}$ coin flip of person I

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is heads. Let B_2 be the event that the $(n+1)^{st}$ coin flip of person I is tails. Then $B_1 \cap B_2 = \emptyset$ since the $(n+1)^{st}$ coin flip of cannot be both heads and tails. And $B_1 \cup B_2 = \Omega$, since the $(n+1)^{st}$ coin flip must be either heads or tails. So, by the total probability theorem,

$$\mathbf{P}(A) = \mathbf{P}(A|B_1)\mathbf{P}(B_1) + \mathbf{P}(A|B_2)\mathbf{P}(B_2).$$

Now, since the $(n+1)^{st}$ coin flip is a fair coin, $\mathbf{P}(B_1) = \mathbf{P}(B_2) = 1/2$. That is,

$$\mathbf{P}(A) = \frac{1}{2}\left(\mathbf{P}(A|B_1) + \mathbf{P}(A|B_2)\right).$$

Given that B_1 occurs, the event A is equal to the event that $S_I \geq S_{II}$. Given that B_2 occurs, the event A is equal to the event $S_I > S_{II}$. So,

$$\mathbf{P}(A) = \frac{1}{2}\left(\mathbf{P}(S_I \geq S_{II}) + \mathbf{P}(S_I > S_{II})\right).$$

Now, $\mathbf{P}(S_I > S_{II}) = \mathbf{P}(S_1 < S_{II})$ by symmetry (with respect to interchanging the roles of person I and person II). So,

$$\mathbf{P}(A) = \frac{1}{2}\left(\mathbf{P}(S_I \geq S_{II}) + \mathbf{P}(S_I < S_{II})\right) = \frac{1}{2}.$$

In the last line, we used that the events $S_I \geq S_{II}$ and $S_I < S_{II}$ are disjoint, and their union is all of Ω, so $\mathbf{P}(S_I \geq S_{II}) + \mathbf{P}(S_I < S_{II}) = 1$.

3. Question 3

Suppose you drive a car with 100 tires. Suppose all of the tires are removed. Then, the mechanic now puts the tires back on the car randomly, so that all arrangements of the tires are equally likely. With what probability will no tire end up in its original position?

Solution. For any $1 \leq i \leq 100$, let A_i be the event that the ith tire ends up in its original position. The union $\cup_{i=1}^{100} A_i$ is the event that at least one tire is put on the original wheel. So, $(\cup_{i=1}^{100} A_i)^c$ is the event that no tire ends up in its original position. From the inclusion-exclusion formula,

$$\mathbf{P}(\cup_{i=1}^{100} A_i) = \sum_{i=1}^{n} \mathbf{P}(A_i) - \sum_{1\leq i<j\leq 100} \mathbf{P}(A_i \cap A_j) + \sum_{1\leq i<j<k\leq 100} \mathbf{P}(A_i \cap A_j \cap A_k)$$
$$\cdots + (-1)^{100+1}\mathbf{P}(A_1 \cap \cdots \cap A_{100}).$$

Fix $1 \leq k \leq 100$ and let $1 \leq i_1 < \cdots < i_k$, then $A_{i_1} \cap \cdots \cap A_{i_k}$ is the event that the tires $i_1, \ldots, i_k$ all end up in their original position. With all of these tire positions fixed, there are $(n-k)!$ permutations of the remaining $n-k$ tires. Since $\mathbf{P}$ is the number of permutations of an event divided by all permutations of the tires (which is 100!), we conclude that

$$\mathbf{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \frac{(100-k)!}{100!}$$

Also, there are $\binom{100}{k}$ ways to choose the indices $1 \leq i_1 < \cdots < i_k \leq 100$ in the k^{th} term in the inclusion-exclusion formula. Therefore

$$\mathbf{P}(\cup_{i=1}^{100} A_i) = 100\frac{(100-1)!}{100!} - \binom{100}{2}\frac{(100-2)!}{100!} + \binom{100}{3}\frac{(100-3)!}{100!} + \cdots + (-1)^{100+1}\frac{1}{100!}$$
$$= \sum_{k=1}^{100}(-1)^{k+1}\binom{100}{k}\frac{(100-k)!}{100!} = \sum_{k=1}^{100}(-1)^{k+1}\frac{100!}{(100-k)!k!}\frac{(100-k)!}{100!} = \sum_{k=1}^{100}(-1)^{k+1}\frac{1}{k!}.$$

So the probability that no tire ends up in its original position is

$$\mathbf{P}((\cup_{i=1}^{100} A_i)^c) = 1 - \mathbf{P}(\cup_{i=1}^{100} A_i) = \sum_{k=0}^{100}(-1)^k\frac{1}{k!} \approx \frac{1}{e}.$$

4. Question 4

Let X_1 be a geometric random variable with parameter p_1. (So, if k is a positive integer, then $\mathbf{P}(X_1 = k) = (1-p_1)^{k-1}p_1$.) Let X_2 be a geometric random variable with parameter p_2. Assume that X_1 and X_2 are independent.

- Compute $\mathbf{P}(X_1 \geq X_2)$.
- Compute $\mathbf{P}(X_1 = X_2)$

(You should simplify your answers as best you can.)

*Solution.*Let k be a positive integer, and let A_k be the event that $X_1 = k$. Note that $\mathbf{P}(X_1 \geq k) = \sum_{j=k}^{\infty}\mathbf{P}(X_1 = j) = \sum_{j=k}^{\infty}(1-p_1)^{j-1}p_1 = (1-p_1)^{k-1}$.

Then $A_k \cap A_{k'} = \emptyset$ if $k \neq k'$, $k, k' \geq 1$, and $\cup_{k=1}^{\infty}A_k$ is equal to the whole sample space. So,

$$\mathbf{P}(X_1 \geq X_2) = \sum_{k=1}^{\infty}\mathbf{P}(\{X_1 \geq X_2\} \cap \{X_2 = k\}) = \sum_{k=1}^{\infty}\mathbf{P}(X_1 \geq k, X_2 = k)$$
$$= \sum_{k=1}^{\infty}\mathbf{P}(X_2 = k)\mathbf{P}(X_1 \geq k)\sum_{k=1}^{\infty}\mathbf{P}(X_2 = k)(1-p_1)^{k-1}$$
$$= \sum_{k=1}^{\infty}[(1-p_2)(1-p_1)]^{k-1}p_2 = \frac{p_2}{1-(1-p_2)(1-p_1)}$$

$$\mathbf{P}(X_1 = X_2) = \sum_{k=1}^{\infty}\mathbf{P}(X_1 = k, X_2 = k) = \sum_{k=1}^{\infty}\mathbf{P}(X_1 = k)\mathbf{P}(X_2 = k)$$
$$= \sum_{k=1}^{\infty}p_1p_2(1-p_1)^{k-1}(1-p_2)^{k-1} = p_1p_2\frac{1}{1-(1-p_1)(1-p_2)}$$

5. Question 5

You have three boxes. The first one contains 1 white and 8 black balls, the second one contains 5 white and 4 black balls, and the last one contains 2 white and 1 black ball. You choose one of these three boxes uniformly at random, and then pick a ball from this box also uniformly at random. What is the probability you pick a white ball?

Solution. For any $i \in \{1,2,3\}$, let A_i be the event that the chosen box is the i^{th} box. Then $\cup_{i=1}^{3} A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for each $i \neq j$, $i, j \in \{1,2,3\}$. Note that $\mathbf{P}(A_i) = 1/3$ for each $i \in \{1,2,3\}$ by assumption. Let W be the event that the white ball is chosen. From the total probability theorem,

$$\mathbf{P}(W) = \sum_{i=1}^{3} \mathbf{P}(W|A_i)\mathbf{P}(A_i) = \frac{1}{3}\sum_{i=1}^{3} \mathbf{P}(W|A_i) = \frac{1}{3}(1/9 + 5/9 + 2/3) = 4/9.$$

6. Question 6

Let $a < b$ be fixed real numbers. Let X be a random variable which is uniformly distributed in the interval $[a,b]$. Compute the mean and variance of X. (As usual, you must show your work to receive credit.)

Solution. $\mathbf{E}X = (b-a)^{-1}\int_a^b x dx = (b-a)^{-1}(1/2)(b^2 - a^2) = (1/2)(b+a)$. $\mathbf{E}X^2 = (b-a)^{-1}\int_a^b x^2 dx = (b-a)^{-1}(1/3)(b^3 - a^3) = (1/3)(b^2 + ab + a^2)$. $\mathrm{var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 = (1/3)(b^2 + ab + a^2) - (1/4)(a^2 + b^2 + 2ab) = (1/12)(a^2 + b^2 - 2ab) = (1/12)(a-b)^2$

7. Question 7

Let $c \in \mathbf{R}$. Let X, Y be continuous random variables with joint density $f_{X,Y}$, where

$$f_{X,Y}(x,y) = \begin{cases} c & \text{, if } 0 \leq y < x < 1 \\ c & \text{, if } 0 \leq x \leq 1 \text{ and } x \leq y < 1 - x \\ 0 & \text{, otherwise.} \end{cases}$$

Find the following three things:

- c
- f_X
- $f_{Y|X}$

Solution. To determine c, we use that $\int_{\mathbf{R}^2} f(x,y)dxdy = 1$. Note that $f = c$ on the region $0 < y < x < 1$. This region is a right triangle with side lengths 1 and 1, so this region has area 1/2. Also, the region $0 \leq x \leq 1$ and $x \leq y \leq 1 - x$ is a right triangle with side lengths $1/\sqrt{2}$ and $1/\sqrt{2}$, so this region has area 1/4. Since f is zero elsewhere, we have

$$1 = \int_{\mathbf{R}^2} f(x,y)dxdy = (1/2 + 1/4)c = (3/4)c.$$

We conclude that $c = 4/3$.

Now, from the definition of marginal, $f_X(x) = \int_{\mathbf{R}} f(x,y)dy$. So,

$$f_X(x) = \begin{cases} 0 & \text{, if } x \notin [0,1] \\ \int_0^{1-x} c dy & \text{, if } 0 \leq x \leq 1/2 \\ \int_0^x c dy & \text{, if } 1/2 \leq x \leq 1 \end{cases} = \begin{cases} 0 & \text{, if } x \notin [0,1] \\ c(1-x) & \text{, if } 0 \leq x \leq 1/2 \\ cx & \text{, if } 1/2 \leq x \leq 1 \end{cases}$$

$$= \begin{cases} 0 & \text{, if } x \notin [0,1] \\ (4/3)(1-x) & \text{, if } 0 \leq x \leq 1/2 \\ (4/3)x & \text{, if } 1/2 \leq x \leq 1. \end{cases}$$

Finally, by the definition of marginal density $f_{Y|X}(y|x) = f(x,y)/f_X(x)$. So,

$$f_{Y|X}(y|x) == \begin{cases} c/f_X(x) & \text{, if } f(x,y) = c \\ 0 & \text{, otherwise} \end{cases} = \begin{cases} c/f_X(x) & \text{, if } 1/2 \leq x \leq 1 \text{ and } 0 \leq y < x < 1 \\ c/f_X(x) & \text{, if } 0 \leq x \leq 1/2 \text{ and } x \leq y < 1-x \\ 0 & \text{, otherwise} \end{cases}$$

$$= \begin{cases} 1/x & \text{, if } 1/2 \leq x \leq 1 \text{ and } 0 \leq y < x < 1 \\ 1/(1-x) & \text{, if } 0 \leq x \leq 1/2 \text{ and } x \leq y < 1-x \\ 0 & \text{, otherwise.} \end{cases}$$

8. Question 8

Let X be a positive discrete random variable. Prove:

$$\mathbf{E}\log(X) \leq \log(\mathbf{E}X).$$

Solution. Let $y = \mathbf{E}X$. For any $x > 0$, let $f(x) = -\log x$, so that $f\colon (0,\infty) \to \mathbf{R}$. Note that f is convex, since $f''(x) = 1/x^2 > 0$ for any $x > 0$. Since f is convex, the function f lies above any of its tangent lines. That is, $f(x) \geq f(y) + f'(y)(x-y)$, for all $x, y > 0$. Since we chose $y = \mathbf{E}X > 0$, we have $f(x) \geq f(\mathbf{E}X) + f'(\mathbf{E}X)(x - \mathbf{E}X)$, for all $x \in \mathbf{R}$. Taking expectation with respect to $x = X$, we have $\mathbf{E}f(X) \geq f(\mathbf{E}X) + f'(\mathbf{E}X)(\mathbf{E}X - \mathbf{E}X) = f(\mathbf{E}X)$. That is, $\mathbf{E}(-\log X) \geq -\log(\mathbf{E}X)$, as desired.

9. Question 9

Let X, Y, Z be independent continuous random variables such that

- $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $\forall\, x \in \mathbf{R}$.
- Y is uniformly distributed in $[-2, 2]$.
- $f_Z(z) = \frac{1}{c} e^{-z^{10}}$, $\forall\, z \in \mathbf{R}$. (Here c is a constant chosen so that $\int_{\mathbf{R}} f_Z(z)dz = 1$.)

Compute $\mathbf{P}(X + Y + Z < 0)$.

Since X, Y, Z are independent, we have $f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z)$ for all $x, y, z \in \mathbf{R}$. Note that f_X, f_Y, f_Z are each even functions. So, using the definition of the joint density, and then changing variables,

$$\begin{aligned} \mathbf{P}(X+Y+Z<0) &= \int_{\{(x,y,z)\in\mathbf{R}^3\colon\, x+y+z<0\}} f_X(x)f_Y(y)f_Z(z)dxdydz \\ &= \int_{\{(x,y,z)\in\mathbf{R}^3\colon\, x+y+z<0\}} f_X(-x)f_Y(-y)f_Z(-z)dxdydz \\ &= \int_{\{(x,y,z)\in\mathbf{R}^3\colon\, -x-y-z<0\}} f_X(x)f_Y(y)f_Z(z)dxdydz \\ &= \int_{\{(x,y,z)\in\mathbf{R}^3\colon\, x+y+z>0\}} f_X(x)f_Y(y)f_Z(z)dxdydz = \mathbf{P}(X+Y+Z>0). \end{aligned}$$

Note that $\mathbf{P}(X+Y+Z=0) = \int_{\{(x,y,z)\in\mathbf{R}^3\colon\, x+y+z=0\}} f_{X,Y,Z}(x,y,z)dxdydz = 0$. So,

$$1 = \mathbf{P}(\Omega) = \mathbf{P}(X+Y+Z<0) + \mathbf{P}(X+Y+Z>0) = 2\mathbf{P}(X+Y+Z<0).$$

That is, $\mathbf{P}(X+Y+Z<0) = 1/2$.

10. Question 10

Let $X_1, \ldots, X_n$ be independent standard Gaussian random variables. Let $Y = \max(X_1, \ldots, X_n)$ be the maximum of $X_1, \ldots, X_n$. Write an integral expression that computes $\mathbf{E}Y$. You should **not** try to evaluate this integral. This integral should be an expression involving the density $e^{-x^2/2}/\sqrt{2\pi}$. (Hint: can you find a relatively simple expression for the CDF of Y?

Solution. As shown in class, the event $Y \leq t$ is equal to the event $X_1 \leq t, \ldots, X_n \leq t$. So, $\mathbf{P}(Y \leq t) = [\mathbf{P}(X_1 \leq t)]^n = [\int_{-\infty}^t e^{-x^2/2} dx/\sqrt{2\pi}]^n$ for any $t \in \mathbf{R}$. We can then get the density of Y, since

$$f_Y(t) = \frac{d}{dt}\mathbf{P}(Y \leq t) = \frac{d}{dt}\left(\int_{-\infty}^t e^{-x^2/2} dx/\sqrt{2\pi}\right)^n = n\left(\int_{-\infty}^t e^{-x^2/2} dx/\sqrt{2\pi}\right)^{n-1} e^{-t^2/2}/\sqrt{2\pi},$$

by the Chain rule and Fundamental Theorem of Calculus. Therefore,

$$\mathbf{E}Y = \int_{-\infty}^{\infty} t f_Y(t) dt = \int_{-\infty}^{\infty} nt \left(\int_{-\infty}^t e^{-x^2/2} dx/\sqrt{2\pi}\right)^{n-1} e^{-t^2/2} dt/\sqrt{2\pi}.$$

11. Question 11

Suppose you have a sequence of integers $a_0, a_1, a_2, \ldots$ such that $a_0 = 0$, $a_1 = 1$, and such that, for any $n \geq 2$, we have $a_n = 2a_{n-1} + 3a_{n-2}$. Using the linear algebraic technique for solving recursions discussed for the Gambler's Ruin problem, find an explicit expression for a_n for any $n \geq 2$.

Solution. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$. Then for any $n \geq 1$,

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}.$$

More generally,

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}. \qquad (*)$$

And A has eigenvalues λ such that $(2-\lambda)(-\lambda) - 3 = 0$, so that $\lambda^2 - 2\lambda - 3 = 0$. From the quadratic formula, or by factoring we have $(\lambda - 3)(\lambda + 1) = 0$, so the eigenvalues are $\lambda = -1$ and $\lambda = 3$. And the eigenvectors of A are in the null space of the respective matrices

$$\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$$

So, two eigenvectors with eigenvalues -1 and 3, respectively, are

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

These eigenvectors are a basis for $\mathbf{R}^2$. Note that $a_1 = 1$ and $a_0 = 0$. We write

$$\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Finally, applying $(*)$ to this equality and using the definition of an eigenvector, we get

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \frac{(-1)^n}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{3^n}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

That is, for any $n \geq 1$,

$$a_n = \frac{1}{4}(-(-1)^n + 3^n).$$

We can test this equality for a few small n's. For example, $a_0 = 0$ and $a_1 = 1$ both still hold.