170A Final Solutions[1]

1. Question 1

(a) There exists a continuous random variable X such that $\mathbf{P}(X = 2) = 1/2$. FALSE. If X is continuous, then $\mathbf{P}(X = a) = 0$ for any $a \in \mathbf{R}$.

(b) There is some random variable X such that $\mathrm{var}(X) = -1$.

FALSE. The variance must be nonnegative by definition.

(c) Let X and Y be random variables. Then

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} \mathbf{E}(X|Y = y) f_X(y) dy.$$

FALSE. The Total Expectation Theorem says $\mathbf{E}(X) = \int_{-\infty}^{\infty} \mathbf{E}(X|Y = y) f_Y(y) dy$.

(d) Let X be a continuous random variable with probability density function f_X. Then $f_X(x) \leq 1$ for all $x \in \mathbf{R}$.

FALSE. A density function can have value larger than 1. For example, if $f_X(x) = 2$ for any $x \in [0, 1/2]$ and $f_X(x) = 0$ otherwise, then f_X is a PDF.

(e) Define $f(x, y) = \begin{cases} 6e^{-3x-2y} & \text{if } x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$. Suppose X and Y are random variables with joint PDF $f(x, y)$. Then X and Y are independent.

TRUE. This follows from Definition 5.53. If $x \geq 0$, then $f_X(x) = \int_0^\infty 6e^{-3x-2y} dy = 3e^{-3x}$ (with $f_X(x) = 0$ otherwise) and if $y \geq 0$, then $f_Y(y) = \int_0^\infty 6e^{-3x-2y} dx = 2e^{-2y}$ (with $f_Y(y) = 0$ otherwise). So, $f_{X,Y}(x, y) = f_X(x) f_Y(y)$ for all $x, y \in \mathbf{R}$. So X and Y are independent.

(f) Let X be a random variable uniformly distributed on the interval $[0, 1]$. Let $Y = -\log X$. (Here log denotes the natural logarithm.) Then Y has CDF given by

$$F_Y(y) = \mathbf{P}(Y \leq y) = \begin{cases} 0 & , y < 0 \\ 1 - e^{-y} & , 0 \leq y. \end{cases}$$

TRUE. Since the logarithm is an increasing function, $\mathbf{P}(Y \leq t) = \mathbf{P}(-\log X \leq t) = \mathbf{P}(\log X \geq -t) = \mathbf{P}(X \geq e^{-t}) = 1 - e^{-t}$. So, the CDF of X is $1 - e^{-y}$ for any $y \geq 0$.

(g) Let X, Y and Z be random variables. Suppose these random variables have joint density function

$$f_{X,Y,Z}(x, y, z) = \begin{cases} \frac{1}{6}(xy + z) & , \text{if } 0 \leq x, y, z \leq 2, \\ 0 & , \text{otherwise.} \end{cases}$$

Then $\mathbf{P}(X \leq 1, Y \leq 1, Z \leq 1) = \frac{1}{8}$.

TRUE. We have $\mathbf{P}(X \leq 1, Y \leq 1, Z \leq 1) = \frac{1}{6} \int_0^1 \int_0^1 \int_0^1 (xy + z) dx dy dz = \frac{1}{6}(1 \cdot (1/2) \cdot (1/2) + 1 \cdot 1 \cdot (1/2)) = (1/6)(3/4) = 1/8$.

(h) Let X be a random variable that only takes nonnegative integer values. Assume that for any integer $n > 10$, we have $\mathbf{P}(X \geq n) = 1/\sqrt{n}$. Then $\mathbf{E}(X) < \infty$.

FALSE. From an exercise from the homework, $\mathbf{E}X = \sum_{n=1}^{\infty} \mathbf{P}(X \geq n)$. But the sum $\sum_{n=10}^{\infty} 1/\sqrt{n}$ diverges. So $EX = \infty$.

[1]June 14, 2017,

(i) For any $x \in \mathbf{R}$, define $F(x) = \frac{\pi}{2} + \tan^{-1}(x)$. Then there exists a random variable X such that $\mathbf{P}(X \leq x) = F(x)$ for all $x \in \mathbf{R}$.

FALSE. If F is a CDF, then $\lim_{x\to\infty} F(x) = 1$. But $\lim_{x\to\infty} F(x) = \pi > 1$.

(j) Let X and Y be random variables on a sample space Ω. Let $\mathbf{P}$ be a probability law on Ω. Assume that X and Y are independent (with respect to the probability law $\mathbf{P}$). Let $\mathbf{P}'$ be another (possibly different) probability law on Ω. Then X and Y are independent, with respect to $\mathbf{P}'$.

FALSE. Let $\Omega = \{0,1\}^2$, let $\mathbf{P}$ be uniform on Ω. Let $X(\omega_1,\omega_2) = \omega_1$ and let $Y(\omega_1,\omega_2) = \omega_2$. Then X and Y are independent (as shown in class). But if $\mathbf{P}'$ is defined by $\mathbf{P}'(\{0,0\}) = \mathbf{P}'(\{0,1\}) = \mathbf{P}'(\{1,0\}) = 1/3$ with $\mathbf{P}'(\{1,1\}) = 0$, then $\mathbf{P}'(X = 0, Y = 0) = 1/3 \neq (2/3)(2/3) = \mathbf{P}'(X = 0)\mathbf{P}'(Y = 0)$.

(k) Let X, Y be random variables. For any $y \in \mathbf{R}$, assume that $\mathbf{E}(X|Y = y) = |y|$. Also, assume that Y is a standard Gaussian random variable. Then $\mathbf{E}(X) = 2$.

FALSE. $\int_{\mathbf{R}} \mathbf{E}(X|Y = y) f_Y = \int_{\mathbf{R}} |y| \, e^{-y^2/2} dy/\sqrt{2\pi} = 2\int_0^\infty y e^{-y^2/2} dy/\sqrt{2\pi} = 2/\sqrt{2\pi} = \sqrt{2/\pi}$.

2. Question 2

Let X be an exponential random variable with parameter $\lambda = 1$. (So, X has density $f_X(x) = e^{-x}$ if $x \geq 0$, and $f_X(x) = 0$ if $x < 0$.) Compute $\mathbf{E}X$ and $\mathbf{E}(X^2)$.

Solution. Integrating by parts, we have $\mathbf{E}X = \int_0^\infty x e^{-x} dx = \int_0^\infty x \frac{d}{dx}(-e^{-x}) dx = \int_0^\infty e^{-x} dx = 1$. Also, $\mathbf{E}X^2 = \int_0^\infty x^2 e^{-x} dx = \int_0^\infty x^2 \frac{d}{dx}(-e^{-x}) dx = \int_0^\infty 2x e^{-x} dx = \int_0^\infty 2e^{-x} dx = 2$.

3. Question 3

Let n be a fixed positive integer. Let X and Y be independent random variables that are uniformly distributed in the set $\{1,\ldots,n\}$. What is the PMF of $X + Y$?

Solution. Let $2 \leq j \leq 2n$ and let $1 \leq k \leq n$. Then $\mathbf{P}(X + Y = j|Y = k) = \mathbf{P}(X = j - k|Y = k) = \mathbf{P}(X = j - k)$, using that X and Y are independent. By the definition of X, $\mathbf{P}(X = j - k) = 1/n$ if $1 \leq j - k \leq n$ and $\mathbf{P}(X = j - k) = 0$ otherwise. So, $\mathbf{P}(X+Y = j|Y = k) = 1/n$ if $1 \leq j-k \leq n$, and $\mathbf{P}(X+Y = j|Y = k) = 0$ otherwise. That is, $\mathbf{P}(X+Y = j|Y = k) = 1/n$ if $k \leq j-1$, $k \geq j-n$, $1 \leq k \leq n$, and $\mathbf{P}(X+Y = j|Y = k) = 0$ otherwise. So, by the Total Probability Theorem, if $2 \leq j \leq 2n$,

$$\mathbf{P}(X + Y = j) = \sum_{k=1}^{n} \mathbf{P}(X + Y = j|Y = k)\mathbf{P}(Y = k) = \sum_{k=\max(1,j-n)}^{\min(n,j-1)} \mathbf{P}(X + Y = j|Y = k)\mathbf{P}(Y = k)$$

$$= \sum_{k=\max(1,j-n)}^{\min(n,j-1)} (1/n)^2 = \begin{cases} (j-1)/n^2 & \text{, if } 2 \leq j \leq n+1 \\ (2n-j+1)/n^2 & \text{, if } n+1 \leq j \leq 2n \end{cases}.$$

For any other j, we have $\mathbf{P}(X + Y = j) = 0$. Consequently, $p_{X+Y}(1) = 0$, $p_{X+Y}(1+(n/3)) = 1/(3n)$, $p_{X+Y}(n+1) = 1/n$, $p_{X+Y}(1+(3n/2)) = 1/(2n)$, $p_{X+Y}(3n) = 0$. (These last answers assume that the arguments of p_{X+Y} are integers.)

4. Question 4

Let X, Y, Z be independent discrete random variables. Prove that X and Y are independent.

Solution. For any $z \in \mathbf{R}$, let $B_z = \{Z = z\}$. Then $B_z \cap B_{z'} = \emptyset$ if $z \neq z'$, $z, z' \in \mathbf{R}$, and $\cup_{z \in \mathbf{R}} B_z = \Omega$. Let $x, y \in \mathbf{R}$. Using Axiom (ii)

$$\mathbf{P}(X = x, Y = y) = \mathbf{P}(\{X = x\} \cap \{Y = y\} \cap (\cup_{z \in \mathbf{R}} B_z)) = \sum_{z \in \mathbf{R}} \mathbf{P}(X = x, Y = y, Z = z). \quad (*)$$

Similarly,

$$\begin{aligned} \mathbf{P}(X = x)\mathbf{P}(Y = y) &= \mathbf{P}(\cup_{z \in \mathbf{R}} B_z)\mathbf{P}(X = x)\mathbf{P}(Y = y) \\ &= \sum_{z \in \mathbf{R}} \mathbf{P}(X = x)\mathbf{P}(Y = y)\mathbf{P}(Z = z) \qquad (**) \end{aligned}$$

By assumption $\mathbf{P}(X = x, Y = y, Z = z) = \mathbf{P}(X = x)\mathbf{P}(Y = y)\mathbf{P}(Z = z)$ for every $x, y, z \in \mathbf{R}$. So, the quantities $(*)$ and $(**)$ are equal for any $x, y \in \mathbf{R}$, as desired.

5. Question 5

A single fair 100-sided die has each of its faces labeled with exactly one integer between and including 1 and 100. Each face is equally like to be rolled.

Suppose you have three fair 100-sided dice. After rolling these three dice, what is the probability that the sum of the rolls of the three dice is 52?

Solution. For any $1 \leq i \leq 100$, let A_i be the event that that first die roll is i. Let B be the event that the sum of the rolls is 52. Then $\mathbf{P}(B) = \sum_{i=1}^{100} \mathbf{P}(B|A_i)\mathbf{P}(A_i)$, by the Total Probability Theorem. (Here we used $\cup_{i=1}^{100} A_i = \Omega$, and $A_i \cap A_j = \emptyset$ for every $i \neq j$ with $1 \leq i, j \leq 100$.) Now, $\mathbf{P}(B|A_i) = 0$ if $i > 50$, since if the first roll exceeds 50, the sum of the rolls must exceed 52, so that $B|A_i$ is empty. So, $\mathbf{P}(B) = \sum_{i=1}^{50} \mathbf{P}(B|A_i)\mathbf{P}(A_i)$. Also, $\mathbf{P}(A_i) = 1/100$ for every $1 \leq i \leq 100$ since the first die is fair, so $\mathbf{P}(B) = \frac{1}{100} \sum_{i=1}^{50} \mathbf{P}(B|A_i)$. Given that A_i occurs, the sum of the remaining two dice is $52 - i = s$. Arguing as in class (or just counting the possibilities), the probability that two of the dice sum to $s = 52 - i$ is $10^{-4}(s - 1) = 10^{-4}(51 - i)$. Therefore,

$$\mathbf{P}(B) = 10^{-6} \sum_{i=1}^{50} (51 - i) = 10^{-6}(50 \cdot 51 - 50 \cdot 51/2) = 10^{-6}(25 \cdot 51) = \frac{51}{40000}.$$

6. Question 6

Let n be a fixed positive integer. Let $X_1, \ldots, X_n$ be independent random variables. As usual, define $\mathrm{var}(X) = \mathbf{E}(X - \mathbf{E}X)^2$. Prove the following:

$$\mathrm{var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{var}(X_i).$$

$$\begin{aligned}
\mathrm{var}(\sum_{i=1}^n X_i) &= \mathbf{E}(\sum_{i=1}^n X_i - \mathbf{E}(\sum_{i=1}^n X_i))^2 = \mathbf{E}(\sum_{i=1}^n (X_i - \mathbf{E}(X_i)))^2 \\
&= \mathbf{E}(\sum_{i=1}^n (X_i - \mathbf{E}(X_i))^2) + 2\mathbf{E}(\sum_{1\leq i<j\leq n} (X_i - \mathbf{E}(X_i))(X_j - \mathbf{E}(X_j))) \\
&= \sum_{i=1}^n \mathrm{var}(X_i) + 2\mathbf{E}(\sum_{1\leq i<j\leq n} \mathbf{E}(X_i - \mathbf{E}(X_i)) \cdot \mathbf{E}(X_j - \mathbf{E}(X_j))) = \sum_{i=1}^n \mathrm{var}(X_i).
\end{aligned}$$

In the penultimate equality, we used that X_i and X_j are independent.

7. Question 7

Let X be binomial random variable with parameters $n = 2$ and $p = 1/2$. So, $\mathbf{P}(X = 0) = 1/4$, $\mathbf{P}(X = 1) = 1/2$ and $\mathbf{P}(X = 2) = 1/4$. And X satisfies $\mathbf{E}X = 1$ and $\mathbf{E}X^2 = 3/2$.

Let Y be a geometric random variable with parameter $1/2$. So, for any positive integer k, $\mathbf{P}(Y = k) = 2^{-k}$. And Y satisfies $\mathbf{E}Y = 4$ and $\mathbf{E}Y^2 = 32$.

Let Z be a Poisson random variable with parameter 1. So, for any nonnegative integer k, $\mathbf{P}(Z = k) = \frac{1}{e}\frac{1}{k!}$. And Z satisfies $\mathbf{E}Z = 1$ and $\mathbf{E}Z^2 = 2$.

Let W be a discrete random variable such that $\mathbf{P}(W = -1) = 2/3$ and $\mathbf{P}(W = 2) = 1/3$, so that $\mathbf{E}W = 0$ and $\mathbf{E}W^2 = 2$.

Assume that X, Y, Z and W are all independent. Compute

$$\mathbf{E}(1 + W^2 + WX^2Y^3Z^4).$$

Solution. From Exercise 4.45 or Remark 4.47 in the notes, since W, X, Y, Z are independent, we have $\mathbf{E}(WX^2Y^3Z^4) = \mathbf{E}(W)\mathbf{E}(X^2)\mathbf{E}(Y^3)\mathbf{E}(Z^4) = 0$, since $\mathbf{E}W = 0$. Therefore, $\mathbf{E}(1 + W^2 + WX^2Y^3Z^4) = 1 + \mathbf{E}W^2 = 1 + 2 = 3$.

8. Question 8

Let X, Y, Z be uniformly distributed random variables on $[0, 1]$. Assume that X, Y and Z are all independent. Compute the probability

$$\mathbf{P}(X + Y + Z < 2).$$

Solution. The joint density is $f_{X,Y,Z}(x, y, z) = 1$ when $0 \leq x, y, z \leq 1$ and $f_{X,Y,Z}(x, y, z) = 0$ otherwise. Since $\mathbf{P}(X + Y + Z < 2) = 1 - \mathbf{P}(X + Y + Z \geq 2)$, we compute the latter probability instead. We then have

$$\begin{aligned}
\mathbf{P}(X + Y + Z \geq 2) &= \iiint_{x+y+z\geq 2} f_{X,Y,Z}(x, y, z)dxdydz = \int_{z=0}^{z=1}\int_{y=1-z}^{1}\int_{x=2-z-y}^{x=1} dxdydz \\
&= \int_{z=0}^{z=1}\int_{y=1-z}^{1} (z + y - 1)dydz = \int_{z=0}^{z=1} [zy + (1/2)y^2 - y]_{y=1-z}^{y=1} dz \\
&= \int_{z=0}^{z=1} [z + 1/2 - 1 - z(1 - z) - (1/2)(1 - z)^2 + (1 - z)]dz \\
&= \int_{z=0}^{z=1} (1/2)z^2 dz = 1/6.
\end{aligned}$$

To determine the integration regions, note that when $x+y+z \geq 2$, when $0 \leq x, y, z \leq 1$ and when z is fixed, we are integrating over the region where $x+y \geq 2-z$, which is a triangular region, with respect to x and y. This region lies between the lines $x+y=2-z$, $y=1$ and $x=1$ in the xy-plane. And the vertices of the triangle are $(1,1-z)$, $(1-z,1)$ and $(1,1)$.

Or, if we knew ahead of time that this integration region was a right triangular pyramid, with edges of length 1, we could compute its volume as $(1/3)(\text{Area of base})(\text{height}) = (1/3)(1/2)(1) = 1/6$.

In any case, $\mathbf{P}(X+Y+Z<2) = 1-\mathbf{P}(X+Y+Z \geq 2) = 1-1/6 = 5/6$.

9. Question 9

Suppose I have a fair coin. So, each coin flip has probability 1/2 of landing heads, and probability 1/2 of landing tails. Suppose I flip the coin ten times, and each time it lands heads. When I flip the coin again an eleventh time, what is the probability that the coin lands heads?

Solution. The probability is 1/2. Each coin flip is independent of the other ones, so any individual coin flip has probability 1/2 of landing heads.

10. Question 10

Let $X_1, \ldots, X_n$ be independent standard Gaussian random variables. Let $Y = \max(X_1, \ldots, X_n)$ be the maximum of $X_1, \ldots, X_n$. Write an integral expression that computes $\mathbf{E}Y$. You should **not** try to evaluate this integral. This integral should be an expression involving the density $e^{-x^2/2}/\sqrt{2\pi}$. (Hint: can you find a relatively simple expression for the CDF of Y?

Solution. As shown in class, the event $Y \leq t$ is equal to the event $X_1 \leq t, \ldots, X_n \leq t$. So, $\mathbf{P}(Y \leq t) = [\mathbf{P}(X_1 \leq t)]^n = [\int_{-\infty}^{t} e^{-x^2/2} dx/\sqrt{2\pi}]^n$ for any $t \in \mathbf{R}$. We can then get the density of Y, since

$$f_Y(t) = \frac{d}{dt}\mathbf{P}(Y \leq t) = \frac{d}{dt}\left(\int_{-\infty}^{t} e^{-x^2/2} dx/\sqrt{2\pi}\right)^n = n\left(\int_{-\infty}^{t} e^{-x^2/2} dx/\sqrt{2\pi}\right)^{n-1} e^{-t^2/2}/\sqrt{2\pi},$$

by the Chain rule and Fundamental Theorem of Calculus. Therefore,

$$\mathbf{E}Y = \int_{-\infty}^{\infty} t f_Y(t) dt = \int_{-\infty}^{\infty} nt\left(\int_{-\infty}^{t} e^{-x^2/2} dx/\sqrt{2\pi}\right)^{n-1} e^{-t^2/2} dt/\sqrt{2\pi}.$$

11. Question 11

Suppose you have a sequence of integers $a_0, a_1, a_2, \ldots$ such that $a_0 = 0$, $a_1 = 1$, and such that, for any $n \geq 2$, we have $a_n = 2a_{n-1} + 3a_{n-2}$. Using the linear algebraic technique for solving recursions discussed for the Gambler's Ruin problem, find an explicit expression for a_n for any $n \geq 2$.

Solution. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$. Then for any $n \geq 1$,

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}.$$

More generally,

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}. \qquad (*)$$

And A has eigenvalues λ such that $(2-\lambda)(-\lambda) - 3 = 0$, so that $\lambda^2 - 2\lambda - 3 = 0$. From the quadratic formula, or by factoring we have $(\lambda-3)(\lambda+1) = 0$, so the eigenvalues are $\lambda = -1$ and $\lambda = 3$. And the eigenvectors of A are in the null space of the respective matrices

$$\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$$

So, two eigenvectors with eigenvalues -1 and 3, respectively, are

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

These eigenvectors are a basis for $\mathbf{R}^2$. Note that $a_1 = 1$ and $a_0 = 0$. We write

$$\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Finally, applying $(*)$ to this equality and using the definition of an eigenvector, we get

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \frac{(-1)^n}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{3^n}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

That is, for any $n \geq 1$,

$$a_n = \frac{1}{4}(-(-1)^n + 3^n).$$

We can test this equality for a few small n's. For example, $a_0 = 0$ and $a_1 = 1$ both still hold.

12. Question 12

Suppose you have a standard 52-card deck of playing cards. (So the cards are sitting in a deck one on top of the other; there are thirteen cards of each of the four suits: hearts, spades, diamonds and clubs. And all permutations of the cards as a single deck of cards are equally likely.) Suppose you are drawing cards from the top of the deck without replacing them, and you put the cards in a pile. What is the expected number of cards you have to draw from the top of the deck before you find **two hearts**? (That is, what is the expected number of cards you have to draw out of the deck right before the pile goes from having one heart to having two hearts?)

Solution. Suppose we label the non-heart cards as $\{1, \ldots, 39\}$. Let $i \in \{1, \ldots, 39\}$. Let $X_i = 1$ if the i^{th} card is drawn before any heart is drawn, and $X_i = 0$ otherwise. Let $Y_i = 1$ if the i^{th} card is drawn between the first heart and the second heart, and $Y_i = 0$ otherwise. The number of cards drawn before the first heart is $\sum_{i=1}^{39} X_i$, and the number of cards drawn after the first heart and before the second heart is $\sum_{i=1}^{39} Y_i$. So, the number of cards drawn before drawing the second heart is

$$1 + \sum_{i=1}^{39}(X_i + Y_i).$$

It remains to compute the expected value of this quantity. We claim that $\mathbf{E}X_i = \mathbf{E}Y_i = 1/14$ for all $i \in \{1, \ldots, 39\}$. Assuming this claim, the expected number of cards to be drawn before the second heart is

$$\mathbf{E}(1 + \sum_{i=1}^{39}(X_i + Y_i)) = 1 + \sum_{i=1}^{39}(\mathbf{E}X_i + \mathbf{E}Y_i) = 1 + 2 \cdot 39/14 = 1 + 39/7 = 46/7.$$

We now prove the claim. Suppose we label the heart at the highest point in the deck as $j = 1$, we label the next highest position heart as $j = 2$ and so on, up to $j = 13$. Then there are fourteen possible locations for a non-heart card: above the $j = 1$ heart, in between the $j = 1$ and $j = 2$ hearts, in between the $j = 2$ and $j = 3$ hearts, etc. For any fixed $i \in \{1, \ldots, 39\}$, the i^{th} card is equally likely to be in any of these 14 locations. To see this, for any of the fourteen $k \in \{1, \ldots, 14\}$ non-heart card locations, let A_k be the event that the i^{th} card is in location k. Then $\cup_{k=1}^{14} A_k = \Omega$ and if $k, k' \in \{1, \ldots, 14\}$ with $k \neq k'$, then $A_k \cap A_{k'} = \emptyset$. Given any arrangement of cards such that the i^{th} card is in location k, we can uniquely associate this arrangement to another arrangement where the i^{th} card occurs in location k'. We can do this, for example, by swapping all cards in location k with all cards in location k'. Since the probability law $\mathbf{P}(A_k)$ counts the number of arrangements in A_k divided by 52!, we conclude that $\mathbf{P}(A_k) = \mathbf{P}(A_{k'})$ for all $k \neq k'$, $k, k' \in \{1, \ldots, 14\}$. So, $1 = \mathbf{P}(\Omega) = \sum_{k=1}^{\infty} \mathbf{P}(A_k) = 14\mathbf{P}(A_1)$. So, $\mathbf{P}(A_1) = \mathbf{P}(A_2) = 1/14$. That is, $\mathbf{P}(X_i = 1) = \mathbf{P}(Y_i = 1) = 1/14$. And since X_i, Y_i only take values 1 or 0, the definition of expected value says $\mathbf{E}X_i = \mathbf{E}Y_i = 1/14$ for all $i \in \{1, \ldots, 39\}$, as desired.