170A Final Solutions[1]

1. Question 1

(a) A continuous random variable is a random variable X together with a continuous function $f_X\colon \mathbf{R}\to\mathbf{R}$ such that, for any $-\infty\leq a<b\leq\infty$, we have $\mathbf{P}(a\leq X\leq b)=\int_a^b f_X(x)dx$.

FALSE. f_X does not have to be continuous. For example, a random variable uniformly distributed on $[0,1]$ has a discontinuous PDF.

(b) There is some random variable X with PDF given by $f_X(x)=x$ for any $x\in[0,2]$, and $f_X(x)=0$ otherwise.

FALSE. The integral of f_X is not 1, so f_X is not a PDF.

(c) Let X and Y be independent random variables taking values in $[-10,10]$. Then

$$\mathbf{E}(XY^2)=\mathbf{E}(X)\mathbf{E}(Y^2).$$

TRUE. This follows from independence and Exercise 4.45. The identity $\mathbf{E}[g(X)h(Y)]=[\mathbf{E}g(X)][\mathbf{E}h(Y)]$ applies when X,Y are independent, so we use $g(x)=x$ and $h(y)=y^2$.

(d) Let $\Omega=\{1,2,3,4,5,6,7\}$. For any subset $A\subseteq\Omega$, let $\mathbf{P}(A)$ be the number of elements of A. Then $\mathbf{P}$ is a probability law on Ω.

FALSE. $\mathbf{P}(\Omega)=7\neq 1$, violating the last axiom for probability laws.

(e) Define $f(x,y)=\begin{cases}12x^2y^3 & \text{if}\, 0\leq x\leq 1,\, 0\leq y\leq 1\\ 0 & \text{otherwise}\end{cases}$. Suppose X and Y are random variables with joint PDF $f(x,y)$. Then X and Y are independent.

TRUE. This follows from Definition 5.53. In particular, note that if $x\in[0,1]$, then $f_X(x)=\int_0^1 12x^2y^3dy=3x^2$ (with $f_X(x)=0$ otherwise) and if $y\in[0,1]$, then $f_Y(y)=\int_0^1 12x^2y^3dx=4y^3$ (with $f_Y(y)=0$ otherwise). So, $f_{X,Y}(x,y)=f_X(x)f_Y(y)$ for all $x,y\in\mathbf{R}$. So X and Y are independent.

(f) Let X,Y be random variables with joint PDF $f_{X,Y}$. Let $x,y\in\mathbf{R}$ with $f_X(x)>0$ and $f_Y(y)>0$. Then

$$f_{X|Y}(x|y)=\frac{f_X(x)f_{Y|X}(y|x)}{\int_{-\infty}^{\infty}f_Y(t)f_{Y|X}(y|t)dt}$$

FALSE. The Continuous Bayes' rule says $f_{X|Y}(x|y)=\frac{f_X(x)f_{Y|X}(y|x)}{\int_{-\infty}^{\infty}f_X(t)f_{Y|X}(y|t)dt}$.

(g) Let X be a random variable uniformly distributed on the interval $[0,1]$. Let $Y=4X(1-X)$. Then the CDF of Y is

$$F_Y(y)=\mathbf{P}(Y\leq y)=\begin{cases}0 & ,y<0\\ 1-\sqrt{1-y} & ,0\leq y\leq 1\\ 1 & ,y>1.\end{cases}$$

TRUE. Using the quadratic formula, the function $f(t)=4t(1-t)$ takes the value $c\in[0,1]$ when $x=(1/2)\pm(1/2)\sqrt{1-c}$. So, if $x\in[0,1]$, we have

$$\begin{aligned}\mathbf{P}(4X(1-X)\leq x)&=\mathbf{P}(X\in[0,1/2-(1/2)\sqrt{1-x}]\text{ or }X\in[1/2+(1/2)\sqrt{1-x},1])\\&=(1/2)-(1/2)\sqrt{1-x}+1-(1/2+(1/2)\sqrt{1-x})=1-\sqrt{1-x}.\end{aligned}$$

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(h) Let X be a random variable uniformly distributed on the interval $[0,1]$. Let $Y = -\log X$. (Here log denotes the natural logarithm.) Then Y has PDF given by

$$f(y) = \begin{cases} 0 & , \text{if } y < 0 \\ e^{-y} & , \text{if } y \geq 0. \end{cases}.$$

TRUE. Since the logarithm is an increasing function, $\mathbf{P}(Y \leq t) = \mathbf{P}(-\log X \leq t) = \mathbf{P}(\log X \geq -t) = \mathbf{P}(X \geq e^{-t}) = 1 - e^{-t}$ if $t \geq 0$. (And $\mathbf{P}(Y \leq t) = 1$ for any $t > 0$.) So, the CDF of X is $1 - e^{-y}$ for any $y \geq 0$. So, the PDF is the derivative of the the CDF, so that the PDF is e^{-y} for any $y \geq 0$.

(i) Let X, Y and Z be random variables. Suppose these random variables have joint density function

$$f_{X,Y,Z}(x,y,z) = \begin{cases} \frac{1}{16}(xy + z) & , \text{if } 0 \leq x, y, z \leq 2, \\ 0 & , \text{otherwise.} \end{cases}$$

Then $\mathbf{E}X = \frac{7}{6}$.

TRUE. We have $\mathbf{E}X = \frac{1}{16}\int_0^2\int_0^2\int_0^2 (x^2y + xz)dxdydz = \frac{1}{16}((1/3)8(2)(2) + (2)(2)(2)) = (1/16)(32/3 + 8) = 56/48 = 7/6$.

(j) Let X be a random variable such that

$$\mathbf{P}(X \leq x) = \begin{cases} 0 & , \text{if } x < 0 \\ x^2 & , \text{if } 0 \leq x \leq 1. \\ 1 & , \text{if } x \geq 1 \end{cases}$$

Then $\mathbf{E}X = \frac{1}{15}$.

FALSE. The density of X is the derivative of the CDF, so the PDF is $2x$ for any $0 \leq x \leq 1$. So, the expected value of X is $\int_0^1 x2xdx = \int_0^1 2x^2dx = 2/3 \neq 1/15$.

(k) Define

$$F(x) = \begin{cases} 0 & , \text{if } x < 0 \\ x^2 - x & , \text{if } 0 \leq x \leq \frac{1+\sqrt{5}}{2} \\ 1 & , \text{if } x \geq \frac{1+\sqrt{5}}{2}. \end{cases}$$

Then there exists a random variable X such that $\mathbf{P}(X \leq x) = F(x)$ for all $x \in \mathbf{R}$

FALSE. $F(1/2) = -1/4$, so it cannot be true that $\mathbf{P}(X \leq 1/2) = -1/4$.

2. Question 2

An urn contains three red cubes and two blue cubes. A cube is removed from the urn uniformly at random. If the cube is red, it is kept out of the urn and a second cube is removed from the urn. If the cube is blue, then this cube is put back into the urn and an additional red cube is put into the urn, and then a second cube is removed from the urn.

- What is the probability that the second cube removed from the urn is red?
- If it is given information that the second cube removed from the urn is red, then what is the probability that the first cube removed from the urn is blue?

Solution. Let A be the event that the first cube removed is red, and let B be the event that the first cube removed is blue. Let C be the event that the second cube removed from the urn is red. Then $A \cap B = \emptyset$ and $A \cup B = \Omega$, so the Total Probability Theorem says

$$\mathbf{P}(C) = \mathbf{P}(C|A)\mathbf{P}(A) + \mathbf{P}(C|B)\mathbf{P}(B) = (1/2)(3/5) + (4/6)(2/5) = 3/10 + 8/30 = 17/30.$$

Now, using that $\mathbf{P}(C) = 17/30$, we have

$$\mathbf{P}(B|C) = \mathbf{P}(C|B)[\mathbf{P}(B)/\mathbf{P}(C)] = (4/6)(2/5)(30/17) = 8/17.$$

3. Question 3

Let $a < b$ be fixed real numbers. Let X be a random variable which is uniformly distributed in the interval $[a, b]$. Compute the mean and variance of X. (As usual, you must show your work to receive credit.)

Solution. $\mathbf{E}X = (b-a)^{-1}\int_a^b x dx = (b-a)^{-1}(1/2)(b^2 - a^2) = (1/2)(b+a)$. $\mathbf{E}X^2 = (b-a)^{-1}\int_a^b x^2 dx = (b-a)^{-1}(1/3)(b^3 - a^3) = (1/3)(b^2 + ab + a^2)$. $\mathrm{var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 = (1/3)(b^2 + ab + a^2) - (1/4)(a^2 + b^2 + 2ab) = (1/12)(a^2 + b^2 - 2ab) = (1/12)(a-b)^2$

4. Question 4

Let X, Y, Z be independent continuous random variables. (That is, $f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z)$ for all $x, y, z \in \mathbf{R}$, and $f_{X,Y,Z}$ is defined so that $\int_s^t \int_c^d \int_a^b f_{X,Y,Z}(x,y,z)dxdydz = \mathbf{P}(a \leq X \leq b, c \leq Y \leq d, s \leq Z \leq t)$ for all $a \leq b$, $c \leq d$, $s \leq t$.)

Assume that X, Y, Z are all uniformly distributed in the interval $[0, 6]$. Prove that X and Y are independent.

Solution 1. Let $x, y \in \mathbf{R}$. Then

$$\begin{aligned}
f_{X,Y}(x,y) &= \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)dz \qquad \text{, by definition of marginal} \\
&= \int_{-\infty}^{\infty} f_X(x)f_Y(y)f_Z(z)dz \qquad \text{, by assumption} \\
&= f_X(x)f_Y(y)\int_{-\infty}^{\infty} f_Z(z)dz = f_X(x)f_Y(y) \qquad \text{, since } \int_{-\infty}^{\infty} f_Z(z)dz = 1.
\end{aligned}$$

That is, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x, y \in \mathbf{R}$, so X and Y are independent.

Solution 2. Let $a \leq b$, $c \leq d$. It is given that $f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z)$ for all $x, y, z \in \mathbf{R}$. So,

$$\begin{aligned}
\int_c^d \int_a^b f_{X,Y}(x,y)dxdy &= \mathbf{P}(a \leq X \leq b, c \leq Y \leq d), \qquad \text{, by definition of } f_{X,Y} \\
&= \mathbf{P}(a \leq X \leq b, c \leq Y \leq d, -\infty \leq Z \leq \infty) \\
&= \int_{-\infty}^{\infty} \int_c^d \int_a^b f_{X,Y,Z}(x,y,z)dxdydz, \qquad \text{, by definition of } f_{X,Y,Z} \\
&= \left(\int_c^d \int_a^b f_X(x)f_Y(y)dxdy\right)\left(\int_{-\infty}^{\infty} f_Z(z)dz\right), \qquad \text{, by assumption} \\
&= \left(\int_c^d \int_a^b f_X(x)f_Y(y)dxdy\right).
\end{aligned}$$

The last line uses $\int_{-\infty}^{\infty} f_Z(z)dz = 1$, which holds since f_Z is a PDF. In conclusion, we have shown $\int_c^d \int_a^b f_{X,Y}(x,y)dxdy = \int_c^d \int_a^b f_X(x)f_Y(y)dxdy$ for all $a \leq b$, $c \leq d$. Since $f_X(x) = 1/6$ if $x \in [0,6]$ and 0 otherwise (and similarly for f_Y), we have (for $0 \leq c \leq d \leq 6$, $0 \leq a \leq b \leq 6$)

$$\int_c^d \int_a^b f_{X,Y}(x,y)dxdy = \frac{1}{36}\int_c^d \int_a^b dxdy = \frac{(b-a)(d-c)}{36}.$$

Since this holds for every $0 \leq c \leq d \leq 6$, $0 \leq a \leq b \leq 6$, we conclude that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x, y \in \mathbf{R}$, as desired. That is, $f_{X,Y}(x,y) = \frac{1}{36}$ if $0 \leq x \leq 6$ and $0 \leq y \leq 6$, and $f_{X,Y}(x,y) = 0$ otherwise.

5. Problem 5

Let $x_1, \ldots, x_n$ be distinct numbers. Consider a random ordered list of the form $y_1, \ldots, y_n$, where the list $y_1, \ldots, y_n$ is a permutation of the numbers $x_1, \ldots, x_n$. Assume that all possible permutations of the numbers $x_1, \ldots, x_n$ are equally likely to occur. For any $i \in \{1, \ldots, n\}$, let A_i be the event that $y_i > y_j$ for all j such that $1 \leq j \leq i-1$. Prove that $\mathbf{P}(A_i) = 1/i$ for any $1 \leq i \leq n$. (As usual, and especially for this problem, you are expected to show all of the details of this proof, and justify every step of your argument.)

Solution 1. Fix $i \in \{1, \ldots, n\}$. Let $j \in \{1, \ldots, i\}$. Let B_j be the event that $a_j > a_k$ for every $k \in \{1, \ldots, i\}$ such that $k \neq j$. Then $\cup_{j=1}^{i} B_j = \Omega$, and $B_j \cap B_{j'} = \emptyset$ for every $j, j' \in \{1, \ldots, i\}$ with $j \neq j'$. So, $1 = \mathbf{P}(\Omega) = \sum_{j=1}^{i} \mathbf{P}(B_j)$. We now claim that $\mathbf{P}(B_j) = \mathbf{P}(B_{j'})$ for every $j, j' \in \{1, \ldots, i\}$ with $j \neq j'$. Given that this is true, it immediately follows that $\mathbf{P}(B_i) = 1/i = \mathbf{P}(A_i)$, as desired. To prove our claim, suppose we denote any ordering of the numbers as $c_1, \ldots, c_n$ where $c_1, \ldots, c_n$ are distinct elements of $\{1, \ldots, n\}$. (That is, we denote the ordering $y_1, \ldots, y_n$ as $x_{c_1}, \ldots, x_{c_n}$.) Then for any $k < i$, any ordering $c_1, \ldots, c_n$ where x_{c_i} exceeds $x_{c_1}, \ldots, x_{c_{i-1}}$ can be uniquely associated to the ordering

$$c_1, \ldots, c_{k-1}, c_i, c_{k+1}, c_{k+2}, \ldots, c_{i-2}, c_{i-1}, c_k, c_{i+1}, \ldots, c_n.$$

That is, we take the original ordering, and swap the i^{th} and k^{th} elements in the ordering. So, the number of orderings where the i^{th} number exceeds the previous ones is equal to the number of orderings where the k^{th} number exceeds the first i numbers (if $k < i$). That is, $\mathbf{P}(A_i) = \mathbf{P}(A_k)$.

Solution 2. $\mathbf{P}(A_i)$ is equal to the number of orderings of the numbers $x_1, \ldots, x_n$ such that the i^{th} number exceeds the previous $i-1$ numbers, all divided by the number of orderings of the numbers. That is, $(n!)\mathbf{P}(A_i)$ is the number of ways of choosing i of the n numbers (which is $\binom{n}{i}$), multiplied by the number of orderings of these i numbers such that the last number is largest (which is $(i-1)!$), multiplied by the number of orderings of the remaining $(n-i)$ numbers (which is $(n-i)!$):

$$\mathbf{P}(A_i) = \frac{(i-1)!(n-i)!\binom{n}{i}}{n!} = \frac{(i-1)!(n-i)!\frac{n!}{(n-i)!i!}}{n!} = \frac{(i-1)!}{i!} = \frac{1}{i}.$$

6. Question 6

Let X be a binomial random variable with parameters $n = 2$ and $p = 1/2$. So, $\mathbf{P}(X = 0) = 1/4$, $\mathbf{P}(X = 1) = 1/2$ and $\mathbf{P}(X = 2) = 1/4$. And X satisfies $\mathbf{E}X = 1$ and $\mathbf{E}X^2 = 3/2$.

Let Y be a geometric random variable with parameter $1/2$. So, for any positive integer k, $\mathbf{P}(Y = k) = 2^{-k}$. And Y satisfies $\mathbf{E}Y = 2$ and $\mathbf{E}Y^2 = 6$.

Let Z be a Poisson random variable with parameter 1. So, for any nonnegative integer k, $\mathbf{P}(Z = k) = \frac{1}{e}\frac{1}{k!}$. And Z satisfies $\mathbf{E}Z = 1$ and $\mathbf{E}Z^2 = 2$.

Let W be a discrete random variable such that $\mathbf{P}(W = 0) = 1/2$ and $\mathbf{P}(W = 4) = 1/2$, so that $\mathbf{E}W = 2$ and $\mathbf{E}W^2 = 8$.

Assume that X, Y, Z and W are all independent. As usual, define $\mathrm{var}(X) = \mathbf{E}[(X-\mathbf{E}X)^2]$. Compute

$$\mathrm{var}(X + Y + Z + W).$$

Solution. Since X, Y, Z and W are independent, Corollary 4.40 in the notes says that $\mathrm{var}(X+Y+Z+W) = \mathrm{var}(X)+\mathrm{var}(Y)+\mathrm{var}(Z)+\mathrm{var}(W) = (3/2)-1^2+6-2^2+2-1^2+8-2^2 = 1/2 + 2 + 1 + 4 = 15/2$.

7. Question 7

Let R and Θ be independent random variables. Assume that Θ is uniform on $(-\pi, \pi)$, and assume that R has the PDF given by

$$f(r) = \begin{cases} re^{-r^2/2} & \text{if } r \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = R\cos\Theta$ and let $Y = R\sin\Theta$.

- Find the joint PDF of (X, Y).
- What is the marginal PDF of X? What is the marginal PDF of Y?
- Are X and Y independent? Explain your reasoning.

Solution 1. Let $f_{R,\Theta}(r, \theta)$ denote the joint PDF of (R, Θ). Since R, Θ are independent, we have $f_{R,\Theta}(r, \theta) = f(r)/2\pi$ whenever $\theta \in (-\pi, \pi)$ and $r \geq 0$, and $f_{R,\Theta}(r, \theta) = 0$ otherwise.

Let A be a subset of the plane $\mathbf{R}^2$. That is, A consists of a set of pairs (x, y) in the plane. Using polar coordinates, we can equivalently think of this set as a set of pairs $(r, \theta) \in A'$, where $x = r\cos\theta$ and $y = r\sin\theta$, and where θ is restricted to $(-\pi, \pi)$. So,

$$\begin{aligned}\mathbf{P}((X, Y) \in A) = \mathbf{P}((R\cos\Theta, R\sin\Theta) \in A) = \mathbf{P}((R, \Theta) \in A') &= \iint_{(r,\theta)\in A'} f_{R,\Theta}(r, \theta)drd\theta \\ = \iint_{(r,\theta)\in A'} f(r)drd\theta = \frac{1}{2\pi}\iint_{(r,\theta)\in A'} re^{-r^2/2}drd\theta \\ = \iint_{(x,y)\in A} \frac{1}{2\pi}e^{-(x^2+y^2)/2}dxdy\end{aligned}$$

So, the joint PDF of X, Y is $f_{X,Y}(x, y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$. The marginal PDFs are

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy\frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx\frac{1}{\sqrt{2\pi}}e^{-y^2/2} = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}.$$

Since $f_{X,Y}(x, y) = f_X(x)f_Y(y)$, the variables X, Y are independent.

Solution 2 for part 1. We could also use an exercise from the last homework. Let $V(r,\theta) = (r\cos\theta, r\sin\theta)$, and let $S(x,y) = (\sqrt{x^2+y^2}, \tan^{-1}(y/x)) = (r,\theta)$. Let $(X,Y) = V(R,\Theta) = (R\cos\Theta, R\sin\Theta)$. Then

$$\begin{aligned}
f_{X,Y}(x,y) &= f_{R,\Theta}(S(x,y))\,|\mathrm{Jac}S(x,y)| \\
&= \frac{1}{2\pi} f(\sqrt{x^2+y^2}) \left|\det\begin{pmatrix} x(x^2+y^2)^{-1/2} & y(x^2+y^2)^{-1/2} \\ \frac{-yx^{-2}}{1+(y/x)^2} & \frac{1/x}{1+(y/x)^2}\end{pmatrix}\right| \\
&= \frac{1}{2\pi}\sqrt{x^2+y^2}e^{-(x^2+y^2)/2}\left|\det\begin{pmatrix} x(x^2+y^2)^{-1/2} & y(x^2+y^2)^{-1/2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{pmatrix}\right| \\
&= \frac{1}{2\pi}\sqrt{x^2+y^2}e^{-(x^2+y^2)/2}(x^2+y^2)^{-1/2} = \frac{1}{2\pi}e^{-(x^2+y^2)/2}.
\end{aligned}$$

8. Question 8

Let X, Z be discrete random variables. Let $A \subseteq \Omega$. Show that $\mathbf{E}(X+Z|A) = \mathbf{E}(X|A) + \mathbf{E}(Z|A)$.

Solution. Let $t \in \mathbf{R}$. Then

$$\begin{aligned}
\mathbf{P}(\{Z+X=t\}\cap A) &= \sum_{z,x\in\mathbf{R}} \mathbf{P}(\{Z+X=t\}\cap\{Z=z\}\cap\{X=x\}\cap A) \\
&= \sum_{z,x\in\mathbf{R}\colon\, z+x=t} \mathbf{P}(\{Z=z\}\cap\{X=x\}\cap A).
\end{aligned}$$

$$\begin{aligned}
&\mathbf{E}(X+Z|A) = \sum_{t\in\mathbf{R}} t p_{X+Z|A}(t) = \sum_{t\in\mathbf{R}} t\mathbf{P}(\{Z+X=t\}\cap A)/\mathbf{P}(A) \\
&= \sum_{t\in\mathbf{R}} t \sum_{z,x\in\mathbf{R}\colon\, z+x=t} \mathbf{P}(\{Z=z\}\cap\{X=x\}\cap A)/\mathbf{P}(A) \\
&= \sum_{z,x\in\mathbf{R}} (z+x)\mathbf{P}(\{Z=z\}\cap\{X=x\}\cap A)/\mathbf{P}(A) \\
&= \sum_{z,x\in\mathbf{R}} z\mathbf{P}(\{Z=z\}\cap\{X=x\}\cap A)/\mathbf{P}(A) + \sum_{z,x\in\mathbf{R}} x\mathbf{P}(\{Z=z\}\cap\{X=x\}\cap A)/\mathbf{P}(A) \\
&= \sum_{z\in\mathbf{R}} z\mathbf{P}(\{Z=z\}\cap A)/\mathbf{P}(A) + \sum_{x\in\mathbf{R}} x\mathbf{P}(\{X=x\}\cap A)/\mathbf{P}(A) = \mathbf{E}(Z|A) + \mathbf{E}(X|A).
\end{aligned}$$

9. Question 9

Using the De Moivre-Laplace Theorem, estimate the probability that $1,000,000$ coin flips of fair coins will result in more than $501,000$ heads. (Some of the following integrals may be relevant: $\int_{-\infty}^0 e^{-t^2/2}dt/\sqrt{2\pi} = 1/2$, $\int_{-\infty}^1 e^{-t^2/2}dt/\sqrt{2\pi} \approx .8413$, $\int_{-\infty}^2 e^{-t^2/2}dt/\sqrt{2\pi} \approx .9772$, $\int_{-\infty}^3 e^{-t^2/2}dt/\sqrt{2\pi} \approx .9987$.)

Solution. Let $X_i = 1$ if the i^{th} coin flip is heads, and let $X_i = 0$ if the i^{th} coin flips is tails. Then $\sum_{i=1}^n X_i$ is the number of heads that have been flipped. The Theorem says $\lim_{n\to\infty}\mathbf{P}\left(\frac{X_1+\cdots+X_n-(1/2)n}{\sqrt{n}\sqrt{1/4}} \leq a\right) = \int_{-\infty}^a e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}$, for any $a \in \mathbf{R}$. So, using $n = 1000000$,

$a = 2$, we have

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_n - (1/2)n}{\sqrt{n}\sqrt{1/4}} \leq a\right) = \mathbf{P}\left(X_1 + \cdots + X_n \leq n/2 + a\sqrt{n}/2\right)$$
$$= \mathbf{P}\left(X_1 + \cdots + X_n \leq 500000 + 1000\right)$$
$$\approx \int_{-\infty}^{2} e^{-t^2/2}\frac{dt}{\sqrt{2\pi}} \approx .9772.$$

So, the probability of more than 501000 heads is roughly $1 - .9772 = .0228$.

10. Question 10

Suppose a needle of length $\ell > 0$ is kept parallel to the ground. The needle is dropped onto the ground with a random position and orientation. The ground has a grid of equally spaced horizontal lines, where the distance between two adjacent lines is $d > 0$. Suppose $\ell < d$. What is the probability that the needle touches one of the lines? (Since $\ell < d$, the needle can touch at most one line.)

Solution. Let x be the distance of the midpoint of the needle from the closest line. Let θ be the acute angle formed by the needle and any horizontal line. The tip of the needle exactly touches the line when $\sin\theta = x/(\ell/2) = 2x/\ell$. So, any part of the needle touches some line if and only if $x \leq (\ell/2)\sin\theta$. Since the needle has a uniformly random position and orientation, we model X, Θ as random variables with joint distribution uniform on $[0, d/2] \times [0, \pi/2]$. That is,

$$f_{X,\Theta}(x,\theta) = \begin{cases} \frac{4}{\pi d}, & x \in [0, d/2] \text{ and } \theta \in [0, \pi/2] \\ 0, & \text{otherwise.} \end{cases}$$

(Note that $\iint_{\mathbf{R}^2} f_{X,\Theta}(x,\theta)dxd\theta = 1$.) And the probability that the needle touches one of the lines is

$$\iint_{0 \leq x \leq (\ell/2)\sin\theta} f_{X,\Theta}(x,\theta)dxd\theta = \int_{\theta=0}^{\theta=\pi/2}\int_{x=0}^{x=(\ell/2)\sin\theta} \frac{4}{\pi d}dxd\theta$$
$$= \frac{2\ell}{\pi d}\int_{\theta=0}^{\theta=\pi/2} \sin\theta d\theta = \frac{2\ell}{\pi d}[-\cos\theta]_{\theta=0}^{\theta=\pi/2} = \frac{2\ell}{\pi d}.$$

Note that $x \leq \ell/2 < d/2$ always, so the set $0 \leq x \leq (\ell/2)\sin\theta$ is still contained in the set $x \in [0, d/2]$.

In particular, when $\ell = d$, the probability is $2/\pi$

11. Question 11

Let $S \subseteq \mathbf{R}^3$ denote the unit ball. That is,

$$S = \{(x,y,z) \in \mathbf{R}^3 \colon x^2 + y^2 + z^2 \leq 1\}.$$

Let X, Y, Z be random variables such that the vector (X, Y, Z) is uniformly distributed in the ball S. Compute the probability

$$\mathbf{P}(X^2 + Y^2 \leq 1/2).$$

Solution. Let $D \subseteq \mathbf{R}^3$ be the subset of the unit ball where $x^2+y^2 \leq 1/2$. We can integrate over D using e.g. cylindrical coordinates as follows

$$\begin{aligned}
\mathbf{P}(X^2+Y^2 \leq 1/2) &= \frac{\iiint_D dV}{\iiint_S dV} = \frac{\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1/\sqrt{2}}\int_{z=-\sqrt{1-r^2}}^{z=\sqrt{1-r^2}} dz r dr d\theta}{4\pi/3} \\
&= \frac{\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1/\sqrt{2}} r\sqrt{1-r^2} dr d\theta}{2\pi/3} \\
&= \frac{\int_{\theta=0}^{\theta=2\pi}\int_{u=1}^{u=1/2} -(u^{1/2}/2) du d\theta}{2\pi/3} \qquad \text{, substituting } u = 1-r^2 \\
&= \frac{\int_{\theta=0}^{\theta=2\pi} [u^{3/2}/3]_{u=1/2}^{u=1} d\theta}{2\pi/3} = \frac{\int_{\theta=0}^{\theta=2\pi} [1-(1/2)^{3/2}] d\theta}{2\pi} = 1-(1/2)^{3/2}.
\end{aligned}$$

12. Question 12

Suppose you have \$100, and you need to come up with \$1000. You are a terrible gambler but you decide you need to gamble your money to get \$1000. For any amount of money M, if you bet $\$M$, then you win $\$M$ with probability .3, and you lose $\$M$ with probability .7. (If you run out of money, you stop gambling, and if you ever have at least \$1000, then you stop gambling.) Consider the following two possible strategies for gambling:

Strategy 1. Bet as much money as you can, up to the amount of money that you need, each time.

Strategy 2. Make a small bet of \$10 each time.

Explain which strategy is better. That is, explain which strategy has a higher probability of getting \$1000.

Solution. Strategy 1 is much better. The probability of reaching \$1000 with consecutive wins is $(.3)^4$, since if you win every time, your sequence of monetary holdings would be: \$100, \$200, \$400, \$800, \$1000. So, with probability at least $(.3)^4$, you will reach \$1000 in winnings. On the other hand, your ability to make it to \$1000 with Strategy 2 is astronomically low. The Gambler's Ruin problem from Example 2.52 in the notes shows that the probability of reaching \$1000 with \$10 bets is the same as: starting with \$10, making \$1 bets, and stopping when you reach \$0 or \$100. The probability of reaching \$100 is

$$\frac{\left(\frac{.7}{.3}\right)^{10}-1}{\left(\frac{.7}{.3}\right)^{100}-1} \leq \frac{3^{10}}{2^{100}} = \frac{3^{10}}{(2^2)^{10}2^{80}} \leq 2^{-80}.$$

And 2^{-80} is much less than $(.3)^4$. That is, Strategy 1 is far superior to Strategy 2.

Alternatively, we could use the De Moivre-Laplace Theorem to get a very rough approximation for the probability of success of strategy 2. If $X_i = 1$ with probability .3 and $X_i = -1$ with probability .7, then $Y_i = (1+X_i)/2$ is a Bernoulli random variable with parameter $p = .3$. The event that Strategy 2 succeeds is (very roughly) the event that $X_1 + \cdots + X_{100} > 90$. This event is the same as $Y_1 + \cdots + Y_{100} > 95$. the probability of failure of Strategy 2 is (using $n = 100$)

$$\begin{aligned}
\mathbf{P}\left(\frac{Y_1 + \cdots + Y_n - pn}{\sqrt{n}\sqrt{p(1-p)}} \leq 14\right) &= \mathbf{P}\left(Y_1 + \cdots + Y_n \leq pn + 14\sqrt{n}\sqrt{p(1-p)}\right) \\
&= \mathbf{P}\left(Y_1 + \cdots + Y_n \leq 30 + 14(10)(.456)\right) \\
&\approx \mathbf{P}\left(Y_1 + \cdots + Y_{100} \leq 95\right) \\
&\approx \int_{-\infty}^{14} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}}
\end{aligned}$$

That is, the probability of success of strategy 2 is roughly

$$\int_{14}^{\infty} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} \leq \int_{14}^{\infty} t e^{-t^2/2} dt = e^{-98}.$$