170A Midterm 2 Solutions[1]

1. Question 1

(a) The number of permutations of the set $\{1, 2, \ldots, n\}$ is $n!$.
TRUE; this was part (a) of Proposition 2.67 in the ntoes.
(b) There is some random variable X such that $\mathrm{var}(X) = -1$.
FALSE. Since $(X - \mathbf{E}X)^2 \geq 0$, we conclude that $\mathrm{var}(X) = \mathbf{E}(X - \mathbf{E}X)^2 \geq \mathbf{E}0 = 0$.
(c) Let X be a random variable that only takes nonnegative integer values. Assume that for any integer $n > 10$, we have $\mathbf{P}(X \geq n) = 1/\sqrt{n}$. Then $\mathbf{E}(X) < \infty$.
FALSE. From an exercise from the homework, $\mathbf{E}X = \sum_{n=1}^{\infty} \mathbf{P}(X \geq n)$. But the sum $\sum_{n=10}^{\infty} 1/\sqrt{n}$ diverges. So $EX = \infty$.
(d) Let X, Y be discrete random variables. Assume that $\mathbf{E}(X|Y = 1) = \mathbf{E}(X|Y = 2) = 2$, and for any integer $k \geq 3$, assume that $\mathbf{E}(X|Y = k) = 0$. Also, assume that $\mathbf{P}(Y = k) = 2^{-k}$ for any integer $k \geq 1$. Then $\mathbf{E}(X) = 2$.
FALSE. From the Total Expectation Theorem, $\mathbf{E}X = \sum_{k=1}^{\infty} \mathbf{P}(Y = k)\mathbf{E}(X|Y = k) = 2(2^{-1}) + 2(2^{-2}) = 1 + 1/2 = 3/2$.

2. Question 2

Let X, Y be random variables with joint PMF $p_{X,Y}$ such that

$$p_{X,Y}(x, y) = \mathbf{P}(X = x, Y = y) = 1/9, \quad \text{for all integers} \quad 1 \leq x, y \leq 3$$

Compute the probabilities of the following events.

- $X > 1$.
- $X + Y \leq 2$.
- $X^2 + Y^2 > 2$.

Solution. First, note that $p_X(x) = \sum_{y=1}^{3} p_{X,Y}(x, y) = 1/3$ for every $1 \leq x \leq 3$, by Proposition 4.17 in the notes. So, $\mathbf{P}(X > 1) = p_X(2) + p_X(3) = 2/3$.
Now, $\mathbf{P}(X + Y < 2) = \mathbf{P}((X, Y) \in \{(1, 1)\}) = p_{X,Y}(1, 1) = 1/9$.
Finally, $\mathbf{P}(X^2+Y^2 > 2) = 1-\mathbf{P}(X^2+Y^2 \leq 2) = 1-\mathbf{P}(X = 1, Y = 1) = 1-p_{X,Y}(1, 1) = 1 - 1/9 = 8/9$.

3. Question 3

Compute the mean and variance of a Poisson random variable X with parameter $\lambda = 1$. (Recall that $\mathbf{P}(X = k) = \frac{1}{e}\frac{1}{k!}$ for any nonnegative integer k.)
Solution. Using the definition of expected value,

$$\mathbf{E}X = \sum_{k=0}^{\infty} \frac{k}{ek!} = \frac{1}{e}\sum_{k=1}^{\infty} \frac{k}{k!} = \frac{1}{e}\sum_{k=1}^{\infty} \frac{1}{(k-1)!} = \frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{k!} = \frac{e}{e} = 1.$$

In the penultimate equality, we used the power series definition of the exponential function ($e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for any $x \in \mathbf{R}$),

[1]May 22, 2017,

Now, using the expected value rule,

$$\mathbf{E}X^2 = \sum_{k=0}^{\infty} \frac{k^2}{ek!} = \frac{1}{e}\sum_{k=1}^{\infty} \frac{k^2}{k!} = \frac{1}{e}\sum_{k=1}^{\infty} \frac{k}{(k-1)!} = \frac{1}{e}\sum_{k=1}^{\infty} \frac{k-1+1}{(k-1)!} = \frac{1}{e}\sum_{k=1}^{\infty} \frac{k-1}{(k-1)!} + \frac{1}{e}\sum_{k=1}^{\infty} \frac{1}{(k-1)!}$$
$$= \frac{1}{e}\sum_{k=2}^{\infty} \frac{k-1}{(k-1)!} + \frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{e}\sum_{k=2}^{\infty} \frac{1}{(k-2)!} + \frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{k!} = 2\frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{k!} = 2.$$

Therefore, by the (alternate) definition of the variance,

$$\mathrm{var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 = 2 - (1)^2 = 1.$$

4. Question 4

Suppose you have \$100, and you need to come up with \$1000. You are a terrible gambler but you decide you need to gamble your money to get \$1000. For any amount of money M, if you bet $\$M$, then you win $\$M$ with probability .3, and you lose $\$M$ with probability .7. (If you run out of money, you stop gambling, and if you ever have at least \$1000, then you stop gambling.) Consider the following two possible strategies for gambling:

Strategy 1. Bet as much money as you can, up to the amount of money that you need, each time.

Strategy 2. Make a small bet of \$10 each time.

Explain which strategy is better. That is, explain which strategy has a higher probability of getting \$1000.

Solution. Strategy 1 is much better. The probability of reaching \$1000 with consecutive wins is $(.3)^4$, since if you win every time, your sequence of monetary holdings would be: \$100, \$200, \$400, \$800, \$1000. So, with probability at least $(.3)^4$, you will reach \$1000 in winnings. On the other hand, your ability to make it to \$1000 with Strategy 2 is astronomically low. The Gambler's Ruin problem from Example 2.52 in the notes shows that the probability of reaching \$1000 with \$10 bets is the same as: starting with \$10, making \$1 bets, and stopping when you reach \$0 or \$100. The probability of reaching \$100 is

$$\frac{\left(\frac{.7}{.3}\right)^{10} - 1}{\left(\frac{.7}{.3}\right)^{100} - 1} \leq \frac{3^{10}}{2^{100}} = \frac{3^{10}}{(2^2)^{10}2^{80}} \leq 2^{-80}.$$

And 2^{-80} is much less than $(.3)^4$. That is, Strategy 1 is far superior to Strategy 2.

5. Question 5

Suppose you have a standard 52-card deck of playing cards. (So the cards are sitting in a deck one on top of the other; there are thirteen cards of each of the four suits: hearts, spades, diamonds and clubs. And all permutations of the cards as a single deck of cards are equally likely.) Suppose you are drawing cards from the top of the deck without replacing them, and you put the cards in a pile. What is the expected number of cards you have to draw from the top of the deck before you find **two hearts**? (That is, what is the expected number of cards you have to draw out of the deck right before the pile goes from having one heart to having two hearts?)

Solution. Suppose we label the non-heart cards as $\{1, \ldots, 39\}$. Let $i \in \{1, \ldots, 39\}$. Let $X_i = 1$ if the i^{th} card is drawn before any heart is drawn, and $X_i = 0$ otherwise. Let $Y_i = 1$

if the i^{th} card is drawn between the first heart and the second heart, and $Y_i = 0$ otherwise. The number of cards drawn before the first heart is $\sum_{i=1}^{39} X_i$, and the number of cards drawn after the first heart and before the second heart is $\sum_{i=1}^{39} Y_i$. So, the number of cards drawn before drawing the second heart is

$$1 + \sum_{i=1}^{39}(X_i + Y_i).$$

It remains to compute the expected value of this quantity. We claim that $\mathbf{E}X_i = \mathbf{E}Y_i = 1/14$ for all $i \in \{1, \ldots, 39\}$. Assuming this claim, the expected number of cards to be drawn before the second heart is

$$\mathbf{E}(1 + \sum_{i=1}^{39}(X_i + Y_i)) = 1 + \sum_{i=1}^{39}(\mathbf{E}X_i + \mathbf{E}Y_i) = 1 + 2 \cdot 39/14 = 1 + 39/7 = 46/7.$$

We now prove the claim. Suppose we label the heart at the highest point in the deck as $j = 1$, we label the next highest position heart as $j = 2$ and so on, up to $j = 13$. Then there are fourteen possible locations for a non-heart card: above the $j = 1$ heart, in between the $j = 1$ and $j = 2$ hearts, in between the $j = 2$ and $j = 3$ hearts, etc. For any fixed $i \in \{1, \ldots, 39\}$, the i^{th} card is equally likely to be in any of these 14 locations. To see this, for any of the fourteen $k \in \{1, \ldots, 14\}$ non-heart card locations, let A_k be the event that the i^{th} card is in location k. Then $\cup_{k=1}^{14} A_k = \Omega$ and if $k, k' \in \{1, \ldots, 14\}$ with $k \neq k'$, then $A_k \cap A_{k'} = \emptyset$. Given any arrangement of cards such that the i^{th} card is in location k, we can uniquely associate this arrangement to another arrangement where the i^{th} card occurs in location k'. We can do this, for example, by swapping all cards in location k with all cards in location k'. Since the probability law $\mathbf{P}(A_k)$ counts the number of arrangements in A_k divided by 52!, we conclude that $\mathbf{P}(A_k) = \mathbf{P}(A_{k'})$ for all $k \neq k'$, $k, k' \in \{1, \ldots, 14\}$. So, $1 = \mathbf{P}(\Omega) = \sum_{k=1}^{\infty} \mathbf{P}(A_k) = 14\mathbf{P}(A_1)$. So, $\mathbf{P}(A_1) = \mathbf{P}(A_2) = 1/14$. That is, $\mathbf{P}(X_i = 1) = \mathbf{P}(Y_i = 1) = 1/14$. And since X_i, Y_i only take values 1 or 0, the definition of expected value says $\mathbf{E}X_i = \mathbf{E}Y_i = 1/14$ for all $i \in \{1, \ldots, 39\}$, as desired.