170A Spring Midterm 2 Solutions[1]

1. Question 1

(a) The number of permutations of the set $\{1, 2, \ldots, n\}$ is $n!$.
TRUE. Proposition 2.67 in the notes
(b) Let X be a nonnegative random variable on a sample space Ω. Assume that X only takes integer values. Prove that

$$\mathbf{E}(X) = \sum_{n=0}^{\infty} \mathbf{P}(X \geq n).$$

FALSE. If $\mathbf{P}(X = 0) = 1$, then $\mathbf{E}(X) = 0 \cdot 1 \neq 1 = \mathbf{P}(X \geq 0)$.
(c) Let X and Y be discrete random variables with joint PMF $p_{X,Y}$. Assume that X and Y take a finite number of values. Then

$$\mathbf{E}(X + Y) = (\mathbf{E}X) + (\mathbf{E}Y).$$

TRUE. Proposition 4.19 in the notes.
(d) Let X and Y be discrete random variables on a sample space Ω. Let $\mathbf{P}$ be a probability law on Ω. Suppose X and Y have joint PMF $p_{X,Y}$. Then, for any $x, y \in \mathbf{R}$, we have

$$p_{X,Y}(x, y) = \mathbf{P}(X = x, Y = y).$$

TRUE. Definition 4.16 in the notes.
(e) Let $p = 2$. Then there exists a random variable X such that, for any integer k such that $0 \leq k \leq 100$, we have

$$\mathbf{P}(X = k) = \binom{100}{k} p^k (1 - p)^{100-k}.$$

FALSE. Since $(1 - p) < 0$, $\mathbf{P}(X = 99) = \binom{100}{2} p^{99}(1 - p) < 0$, which cannot be true, since $\mathbf{P}(A) \geq 0$ for any event A.

2. Question 2

Let X be a random variable such that, for any positive integer $1 \leq k \leq 500$, we have $p_X(k) = 1/500$. Compute the probabilities of the following events

- $X > 100$.
- $X^4 < 20$.
- $\sin(\pi X/2) = 0$.

Solution. $\mathbf{P}(X > 100) = \sum_{k=101}^{500} p_X(k) = \frac{1}{500}\sum_{k=101}^{500} = \frac{400}{500} = \frac{4}{5}$.
If $X \geq 3$, then $X^4 \geq 81 > 20$. Also, $1^4 < 20$ and $2^4 < 20$. So, $\{X^4 < 20\} = \{1 \leq X \leq 2\}$, since X only takes integer values from 1 to 500 with positive probability. So, $\mathbf{P}(X^4 < 20) = p_X(1) + p_X(2) = 2/500 = 1/250$.
Once again, X only takes integer values from 1 to 500 with positive probability. So, $\{\sin(\pi X/2) = 0\} = \{X \text{ is even}\}$. So,

$$\mathbf{P}(\sin(\pi X/2) = 0) = \sum_{k \in \{2,4,6,\ldots,500\}} p_X(k) = \frac{1}{500} \sum_{k \in \{2,4,6,\ldots,500\}} = \frac{250}{500} = \frac{1}{2}.$$

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3. Question 3

Suppose there are five separate bins. You first place a sphere randomly in one of the bins, where each bin has an equal probability of getting the sphere. Once again, you randomly place another sphere uniformly at random in one of the bins. This process occurs thirty times, so that thirty spheres have been placed in bins. What is the expected number of empty bins at the end? (All of the sphere placements are independent of each other.)

Solution. For each $1 \leq i \leq 5$, let $X_i = 1$ if the i^{th} bin has at least one sphere at the end of the thirty sphere placements, and $X_i = 0$ otherwise. If $1 \leq i \leq 5$ is fixed, then at each step of this process, the i^{th} bin has probability $1 - 1/5$ of not receiving the sphere from that step. This process is repeated 30 times, and each sphere placement is independent of the others, so $\mathbf{P}(X_i = 0) = (1 - 1/5)^{30} = (4/5)^{30}$. So, $\mathbf{P}(X_i = 1) = 1 - \mathbf{P}(X_i = 0) = 1 - (4/5)^{30}$. And $\mathbf{E}X_i = 0 \cdot \mathbf{P}(X_i = 0) + 1 \cdot \mathbf{P}(X_i = 1) = 1 - (4/5)^{30}$. The number of bins with a sphere at the end of the sphere placements is $\sum_{i=1}^{5} X_i$. So, the number of empty bins at the end of the sphere placements is $5 - \sum_{i=1}^{5} X_i$. And the expected value of this quantity is

$$\mathbf{E}(5 - \sum_{i=1}^{5} X_i) = 5 - \sum_{i=1}^{5} \mathbf{E}(X_i) = 5 - 5 + 5(4/5)^{30} = 5(4/5)^{30}.$$

4. Question 4

Let X be a discrete random variable. Prove:

$$\mathbf{E}(e^X) \geq e^{\mathbf{E}X}.$$

Solution. Let $y = \mathbf{E}X$. For any $x \in \mathbf{R}$, let $f(x) = e^x$, so that $f\colon \mathbf{R} \to \mathbf{R}$. Note that f is convex, since $f''(x) = e^x > 0$ for any $x \in \mathbf{R}$. Since f is convex, the function f lies above any of its tangent lines. That is, $f(x) \geq f(y) + f'(y)(x - y)$, for all $x \in \mathbf{R}$. Since we chose $y = \mathbf{E}X$, we have $f(x) \geq f(\mathbf{E}X) + f'(\mathbf{E}X)(x - \mathbf{E}X)$, for all $x \in \mathbf{R}$. Taking expectation with respect to $x = X$, we have $\mathbf{E}f(X) \geq f(\mathbf{E}X) + f'(\mathbf{E}X)(\mathbf{E}X - \mathbf{E}X) = f(\mathbf{E}X)$. That is, $\mathbf{E}e^X \geq e^{\mathbf{E}X}$, as desired.

5. Question 5

Let $0 < p < 1$. Suppose you have a biased coin which has a probability p of landings heads, and probability $1 - p$ of landing tails, each time it is flipped. Also, suppose you have a fair five-sided die (so each face of the cube has a distinct label from the set $\{1, 2, 3, 4, 5\}$, and each time you roll the die, any face of the die is rolled with equal probability.) Similarly, suppose you have a fair seven-sided die and a fair nine-sided die.

Let N be the number of coin flips you need to do until the first head appears. If $N = 2$, roll the fair nine-sided die N times. If N is odd, roll the fair seven-sided die N times. If N is even and $N > 2$, then roll the fair five-sided die N times.

Let S be the sum of the results of the N rolls of the die. Compute $\mathbf{E}S$.

Solution. Let n be a fixed positive integer. Conditioning on the event $N = n$, S has expected value n times the expected value of a single die roll. For the five, seven, and nine-sided dice, the expected value of a single die roll is $3, 4$ and 5, respectively. So,

$$\mathbf{E}(S|N=n) = \begin{cases} 5n & \text{, if } n = 2 \\ 4n & \text{, if } n \text{ is odd} \\ 3n & \text{, if } n > 2 \text{ is even} \end{cases}$$

Then, using the total expectation theorem,

$$\begin{aligned} \mathbf{E}S &= \sum_{n \in \mathbf{R}:\, p_N(n)>0} p_N(n)\mathbf{E}(S|N=n) = \sum_{n=1}^{\infty} p_N(n)\mathbf{E}(S|N=n) \\ &= 10(1-p)p + \sum_{n \geq 1,\, \text{odd}} (1-p)^{n-1}p(4)n + \sum_{n>2,\, \text{even}} (1-p)^{n-1}p(3)n \\ &= 10(1-p)p + \sum_{n=1}^{\infty}(1-p)^{2n-2}p(4)(2n-1) + \sum_{n=2}^{\infty}(1-p)^{2n-1}p(3)2n \end{aligned}$$