170A Midterm 2 Solutions[1]

1. Question 1

(a) The number of ways to make an ordered list of k elements of the set $\{1, 2, \ldots, n\}$ is $n!/(n-k)! = n(n-1)\cdots(n-k+1)$.

TRUE. Proposition 2.66 in the notes.

(b) Let n be a positive integer. Let Ω be a discrete sample space, and let $\mathbf{P}$ be a probability law on Ω. Let $A_1, \ldots, A_n \subseteq \Omega$. Then:

$$\mathbf{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbf{P}(A_i) + \sum_{1 \leq i < j \leq n} \mathbf{P}(A_i \cap A_j).$$

FALSE. This statement for $n = 2$ says $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(A \cap B)$. So, if $A = B = \Omega$, this says $1 = 3$.

(c) Let $\lambda > 0$. For each positive integer n, let $0 < p_n < 1$, and let X_n be a binomial distributed random variable with parameters n and p_n. Assume that $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} np_n = \lambda$. Then, for any nonnegative integer k, we have

$$\lim_{n\to\infty} \mathbf{P}(X_n = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

TRUE. Proposition 3.14 from the notes

(d) Let X be a discrete random variable. Let $\mathrm{var}(X) = \mathbf{E}(X - \mathbf{E}(X))^2$. Let a, b be arbitrary constants. Then

$$\mathrm{var}(aX + b) = a^2 \mathrm{var}(X) + b^2.$$

FALSE. Setting $a = 0, b = 1$, this says the variance of the constant function is 1, which is false (it's variance is zero). The correct statement is $\mathrm{var}(aX + b) = a^2 \mathrm{var}(X)$.

(e) Let X be a nonnegative random variable on a sample space Ω. Assume that X only takes integer values. Prove that

$$\mathbf{E}(X) = \sum_{n=1}^{\infty} \mathbf{P}(X \geq n).$$

TRUE. Exercise 4.12 in the notes.

2. Question 2

Let X be a random variable. Let $Y = |X|$. Assume the PMF of X is given by

$$p_X(x) = \begin{cases} kx^2 & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise,} \end{cases}$$

where k is some fixed constant number.

- Determine the value of k.
- Find the PMF of Y.

[1] February 22, 2016,

Solution. We have $1 = \sum_{k=-3}^{3} p_X(k) = k(9+4+1+0+1+4+9) = 28k$. So, $k = 1/28$ by Proposition 3.8 in the notes. Also, $p_Y(x) = \mathbf{P}(|X| = x)$, and $|X|$ only takes the values $0, 1, 2, 3$ with positive probability. So, $\mathbf{P}(Y = 0) = \mathbf{P}(X = 0) = 0$, $\mathbf{P}(Y = 1) = \mathbf{P}(X = 1 \,\text{or}\, X = -1) = 2/28 = 1/14$, $\mathbf{P}(Y = 2) = \mathbf{P}(X = 2 \,\text{or}\, X = -2) = 8/28 = 2/7$, and $\mathbf{P}(Y = 3) = \mathbf{P}(X = 3 \,\text{or}\, X = -3) = 18/28 = 9/14$. That is, $p_Y(0) = 0$, $p_Y(1) = 1/14$, $p_Y(2) = 2/7$, $p_y(3) = 9/14$, and $p_Y(y) = 0$ for any other y.

3. Question 3

You are trapped in a maze. Your starting point is a room with three doors. The first door will lead you to a corridor which lets you exit the maze after three hours of walking. The second door leads you through a corridor which puts you back to the starting point of the maze after seven hours of walking. The third door leads you through a corridor which puts you back to the starting point of the maze after nine hours of walking. Each time you are at the starting point, you choose one of the three doors with equal probability.

Let X be the number of hours it takes for you to exit the maze. Let Y be the number of the door that you initially choose.

- Compute $\mathbf{E}(X|Y = i)$ for each $i \in \{1, 2, 3\}$, in terms of $\mathbf{E}X$.
- Compute $\mathbf{E}X$.

Solution. $\mathbf{E}(X|Y = 1) = 3$, $\mathbf{E}(X|Y = 2) = 7 + \mathbf{E}X$, and $\mathbf{E}(X|Y = 3) = 9 + \mathbf{E}X$. To see the first equality, note that if $Y = 1$, then you exit the maze in three hours, so $p_{X|Y}(x, 1) = 1$ when $x = 3$, so $\mathbf{E}(X|Y = 1) = 3$. For the second equality, if $Y = 2$, then $p_{X|Y}(x, 2) = p_X(x + 7)$ for all real x. Finally, from the Total Expectation Theorem, $\mathbf{E}X = \sum_{i=1}^{3} p_Y(i)\mathbf{E}(X|Y = i) = (1/3)(3) + (1/3)(7 + \mathbf{E}X) + (1/3)(9 + \mathbf{E}X)$. That is, $\mathbf{E}X = 19/3 + (2/3)\mathbf{E}X$. So, $\mathbf{E}X = 19$.

4. Question 4

Let $b_1, \ldots, b_n$ be distinct numbers, representing the quality of n people. Suppose n people arrive to interview for a job, one at a time, in a random order. That is, every possible arrival order of these people is equally likely. We can think of an arrival ordering of the people as an ordered list of the form $a_1, \ldots, a_n$, where the list $a_1, \ldots, a_n$ is a permutation of the numbers $b_1, \ldots, b_n$. Moreover, we interpret a_1 as the rank of the first person to arrive, a_2 as the rank of the second person to arrive, and so on. And all possible permutations of the numbers $b_1, \ldots, b_n$ are equally likely to occur.

For each $i \in \{1, \ldots, n\}$, upon interviewing the i^{th} person, if $a_i > a_j$ for all $1 \leq j < i$, then the i^{th} person is hired. That is, if the person currently being interviewed is better than the previous candidates, she will be hired. What is the expected number of hirings that will be made?

Solution. Let $X_i = 1$ if the i^{th} person to arrive is hired, and let $X_i = 0$ otherwise. Person 1 will always be hired, i.e. $\mathbf{P}(X_1 = 1) = 1$, so $\mathbf{E}X_1 = 1$. Since any arrival order is equally likely, $\mathbf{P}(X_2 = 1) = 1/2$. So, $\mathbf{E}X_2 = 1/2$. In general, if i is a positive integer, then $\mathbf{P}(X_i = 1) = 1/i$. This follows since any ordering of the people is equally likely, so there is a probability of $1/i$ of the i^{th} person having the largest number a_i among the numbers $a_1, \ldots, a_i$. So, $\mathbf{E}X_i = 1/i$. (More formally, fix $i \in \{1, \ldots, n\}$, and let $j \in \{1, \ldots, i\}$. Let A_j be the event that $a_j > a_k$ for every $k \in \{1, \ldots, i\}$ such that $k \neq j$. Then $\cup_{j=1}^{i} A_j = \Omega$, and $A_j \cap A_{j'} = \emptyset$ for every $j, j' \in \{1, \ldots, i\}$ with $j \neq j'$. So, $1 = \mathbf{P}(\Omega) = \sum_{j=1}^{i} \mathbf{P}(A_j)$. We now claim that $\mathbf{P}(A_j) = \mathbf{P}(A_{j'})$ for every $j, j' \in \{1, \ldots, i\}$ with $j \neq j'$. Given that this is true, it immediately follows that $\mathbf{P}(A_i) = 1/i$, as desired. To prove our claim, suppose we write any arrival order of the people as $c_1, \ldots, c_n$ where $c_1, \ldots, c_n$ are distinct elements of $\{1, \ldots, n\}$. Then for any $k < i$, any arrival order $c_1, \ldots, c_n$ where a_{c_i} exceeds $a_{c_1}, \ldots, a_{c_{i-1}}$ can be uniquely associated to the arrival order $c_1, \ldots, c_{k-1}, c_i, c_{k+1}, \ldots, c_{i-1}, c_k, c_{i+1}, \ldots, c_n$. That is, the number of orderings where the i^{th} number exceeds the previous ones is equal to the number of orderings where the k^{th} number exceeds the first i numbers. That is, $\mathbf{P}(A_i) = \mathbf{P}(A_k)$.)

5. Question 5

Let $0 < p < 1$. Suppose you have a biased coin which has a probability p of landings heads, and probability $1 - p$ of landing tails, each time it is flipped. Also, suppose you have a fair six-sided die (so each face of the cube has a distinct label from the set $\{1, 2, 3, 4, 5, 6\}$, and each time you roll the die, any face of the cube is rolled with equal probability.)

Let N be the number of coin flips you need to do until the first head appears. Now, roll the fair die N times. Let S be the sum of the results of the N rolls of the die. Compute $\mathbf{E}S$.

Solution. Let n be a fixed positive integer. Conditioning on the event $N = n$, S has expected value n times the expected value of a single die roll, which is 7/2. So,

$$\mathbf{E}(S|N = n) = (7/2)n$$

Then, using the total expectation theorem,

$$\mathbf{E}S = \sum_{n \in \mathbf{R}\colon\, p_N(n) > 0} p_N(n)\mathbf{E}(S|N = n) = \sum_{n=1}^{\infty} (1 - p)^{n-1} p(7/2)n = \frac{7}{2} \sum_{n=1}^{\infty} n(1 - p)^{n-1} p.$$

For any $p \in \mathbf{R}$, define $f\colon \mathbf{R} \to \mathbf{R}$ by $f(p) = \sum_{n=1}^{\infty} (1 - p)^n p = p(1 - p)/p = (1 - p)$. Then $f'(p) = -\sum_{n=1}^{\infty} n(1 - p)^{n-1} p + \sum_{n=1}^{\infty} (1 - p)^n = -1$. That is, $\sum_{n=1}^{\infty} n(1 - p)^{n-1} p = 1 + (1 - p)/p = 1/p$, so that

$$\mathbf{E}S = \frac{7}{2}\frac{1}{p}.$$