Probability, 170A, Winter 2016, UCLA — Instructor: Steven Heilman

Name: ________________ UCLA ID: ____________ Date: __________

Signature: ________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck![a]

1. Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

   (a) (2 points) The number of ways to make an ordered list of k elements of the set $\{1, 2, \ldots, n\}$ is $n!/(n-k)! = n(n-1)\cdots(n-k+1)$.

   TRUE FALSE (circle one)

   (b) (2 points) Let n be a positive integer. Let Ω be a discrete sample space, and let $\mathbf{P}$ be a probability law on Ω. Let $A_1, \ldots, A_n \subseteq \Omega$. Then:

   $$\mathbf{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mathbf{P}(A_i) + \sum_{1 \leq i < j \leq n} \mathbf{P}(A_i \cap A_j).$$

   TRUE FALSE (circle one)

   (c) (2 points) Let $\lambda > 0$. For each positive integer n, let $0 < p_n < 1$, and let X_n be a binomial distributed random variable with parameters n and p_n. Assume that $\lim_{n\to\infty} p_n = 0$ and $\lim_{n\to\infty} np_n = \lambda$. Then, for any nonnegative integer k, we have

   $$\lim_{n\to\infty} \mathbf{P}(X_n = k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

   TRUE FALSE (circle one)

   (d) (2 points) Let X be a discrete random variable. Let $\mathrm{var}(X) = \mathbf{E}\left[(X - \mathbf{E}(X))^2\right]$. Let a, b be arbitrary constants. Then $\mathrm{var}(aX + b) = a^2\mathrm{var}(X) + b^2$.

   TRUE FALSE (circle one)

   (e) (2 points) Let X be a nonnegative random variable on a sample space Ω. Assume that X only takes integer values. Then $\mathbf{E}(X) = \sum_{n=1}^\infty \mathbf{P}(X \geq n)$.

   TRUE FALSE (circle one)

2. (10 points) Let X be a random variable. Let $Y = |X|$. Assume the PMF of X is given by

$$p_X(x) = \begin{cases} kx^2 & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise,} \end{cases}$$

where k is some fixed constant number.

- Determine the value of k.
- Find the PMF of Y.

3. (10 points) You are trapped in a maze. Your starting point is a room with three doors. The first door will lead you to a corridor which lets you exit the maze after three hours of walking. The second door leads you through a corridor which puts you back to the starting point of the maze after seven hours of walking. The third door leads you through a corridor which puts you back to the starting point of the maze after nine hours of walking. Each time you are at the starting point, you choose one of the three doors with equal probability.

   Let X be the number of hours it takes for you to exit the maze. Let Y be the number of the door that you initially choose.

   - Compute $\mathbf{E}(X|Y = i)$ for each $i \in \{1, 2, 3\}$, in terms of $\mathbf{E}X$.
   - Compute $\mathbf{E}X$. (Your answer here should be some number.)

4. (10 points) Let $b_1, \ldots, b_n$ be distinct numbers, representing the quality of n people. Suppose n people arrive to interview for a job, one at a time, in a random order. That is, every possible arrival order of these people is equally likely. We can think of an arrival ordering of the people as an ordered list of the form $a_1, \ldots, a_n$, where the list $a_1, \ldots, a_n$ is a permutation of the numbers $b_1, \ldots, b_n$. Moreover, we interpret a_1 as the rank of the first person to arrive, a_2 as the rank of the second person to arrive, and so on. And all possible permutations of the numbers $b_1, \ldots, b_n$ are equally likely to occur.

   For each $i \in \{1, \ldots, n\}$, upon interviewing the i^{th} person, if $a_i > a_j$ for all $1 \leq j < i$, then the i^{th} person is hired. That is, if the person currently being interviewed is better than the previous candidates, she will be hired. What is the expected number of hirings that will be made?

5. (10 points) Let $0 < p < 1$. Suppose you have a biased coin which has a probability p of landings heads, and probability $1 - p$ of landing tails, each time it is flipped. Also, suppose you have a fair six-sided die (so each face of the cube has a distinct label from the set $\{1, 2, 3, 4, 5, 6\}$, and each time you roll the die, any face of the cube is rolled with equal probability.)

   Let N be the number of coin flips you need to do until the first head appears. Now, roll the fair die N times. Let S be the sum of the results of the N rolls of the die. Compute $\mathbf{E}S$.

(Scratch paper)