Please provide complete and well-written solutions to the following exercises.

Due May 10, in the discussion section.

Homework 5

Exercise 1. Suppose we have a two-person zero-sum game with $(n+1)\times(n+1)$ payoff matrix A such that at least one entry of A is nonzero. Let $x,y\in\Delta_{n+1}$. Write $x=(x_1,\ldots,x_n,1-\sum_{i=1}^n x_i)$, $y=(x_{n+1},x_{n+2},\ldots,x_{2n},1-\sum_{i=n+1}^{2n}x_i)$. Consider the function $f\colon \mathbf{R}^{2n}\to\mathbf{R}$ defined by $f(x_1,\ldots,x_{2n})=x^TAy$. Show that the Hessian of f has at least one positive eigenvalue, and at least one negative eigenvalue. Conclude that any critical point of f is a saddle point. That is, if we find a critical point of f (as we sometimes do when we look for the value of the game), then this critical point is a saddle point of f. In this sense, the minimax value occurs at a saddle point of f.

(Hint: Write f in the form $f(x_1,\ldots,x_{2n})=\sum_{i=1}^{2n}b_ix_i+\sum_{\substack{1\leq i\leq n,\\ n+1\leq j\leq 2n}}c_{ij}x_ix_j$, where $b_i,c_{ij}\in\mathbf{R}$. From here, it should follow that there exists a nonzero matrix C such that the Hessian of f, i.e. the matrix of second order partial derivatives of f, should be of the form $\begin{pmatrix}0 & C\\ C^T & 0\end{pmatrix}$. For simplicity, you are allowed to assume that C is invertible. (This assumption makes the exercise easier, since you should be able to show that the determinant of the Hessian is negative, but the assumption that C is invertible is not actually necessary to complete the exercise.))

Exercise 2. Suppose we have a two-person zero-sum game. Show that any optimal strategy is a Nash equilibrium. Then, show that any Nash equilibrium is an optimal strategy. In summary, the Nash equilibrium generalizes the notion of optimal strategy. (Hint: to prove that a Nash equilibrium is an optimal strategy it may be helpful to argue by contradiction, and to assume that there is a Nash equilibrium that is not an optimal strategy. Then, it may be helpful to use the first part of the argument in our proof of the Minimax Theorem.)

Exercise 3. Show that, in any two-player general-sum game, for any $i\in\{1,2\}$, the payoffs for player i in any Nash equilibrium exceeds the minimax value for player i. (If A is the $m\times n$ payoff matrix for player 1, then the minimax value for player 1 is the quantity $\max_{x\in\Delta_m}\min_{y\in\Delta_n}x^TAy=\min_{y\in\Delta_n}\max_{x\in\Delta_m}x^TAy$; If B is the $m\times n$ payoff matrix for player 2, then the minimax value for player 2 is the quantity $\max_{y\in\Delta_m}\min_{x\in\Delta_n}x^TBy=\min_{x\in\Delta_n}\max_{y\in\Delta_m}x^TBy$.)

Exercise 4. Recall the prisoner's dilemma, which is described by the following payoffs

Recall that this two-person game has exactly one Nash equilibrium, where both parties confess. However, if this game is repeated an infinite number of times, or a random number of times, this strategy is no longer the only Nash equilibrium. This exercise explores the case where the game is repeated an infinite number of times. Let N be a positive integer.

Prisoner I	Prisoner II: silent	Prisoner II: confess
silent	$(-1,-1)$	$(-10,0)$
confess	$(0,-10)$	$(-8,-8)$

Suppose the game is repeated infinitely many times, so that player I has payoffs $a_1, a_2, a_3, \ldots$ and player II has payoffs $b_1, b_2, b_3, \ldots$. That is, at the i^{th} iteration of the game, player I has payoff a_i and player II has payoff b_i. In the infinitely repeated game, each player would like to maximize her average payoff over time (if this average exists). That is, player I wants to maximize $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} a_i$ and player II wants to maximize $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} b_i$.

Consider the following strategy for player I. Suppose player I begins by staying silent, and she continues to be silent on subsequent rounds of the game. However, if player II confesses at round $i \geq 1$ of the game, then player I will always confess for every round of the game after round i. Player II follows a similar strategy. Suppose player II begins by staying silent, and she continues to be silent on subsequent rounds of the game. However, if player I confesses at round $j \geq 1$ of the game, then player II will always confess for every round of the game after round j.

Show that this pair of strategies is a Nash equilibrium. That is, no player can gain something by unilaterally deviating from this strategy.

Exercise 5. Show that the following strategy (known as "quid pro quo") is also a Nash equilibrium for Prisoner's Dilemma iterated an infinite number of times.

Player I begins by staying silent. If Player II plays x on round i, then Player I plays x on round $i+1$. Similarly, Player II begins by staying silent. If Player I plays x on round i, then Player II plays x on round $i+1$.

Exercise 6. Find all Correlated Equilibria for the Prisoner's Dilemma.

Exercise 7. We return now to the setting of general sum games. Show that any convex combination of Nash equilibria is a Correlated Equilibrium. That is, if $z(1),\ldots,z(k)$ are Nash Equilibria, and if $t_1,\ldots,t_k \in [0,1]$ satisfy $\sum_{i=1}^{k} t_i = 1$, then $\sum_{i=1}^{k} t_i z(i)$ is a Correlated Equilibrium.