Please provide complete and well-written solutions to the following exercises.

Due May 3, in the discussion section.

Homework 4

Exercise 1. Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 0 & 9 & 1 & 1 \\ 5 & 0 & 6 & 7 \\ 2 & 4 & 3 & 3 \end{pmatrix}$$

Exercise 2. Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 0 & 7 & 0 & 6 \\ 4 & 4 & 3 & 3 \\ 8 & 2 & 6 & 0 \end{pmatrix}$$

Exercise 3. This Exercise shows that von Neumann's Minimax Theorem no longer holds when we consider games for three or more players.

first, note that there is a suitable generalization of this theorem to two-player general-sum games. That is if A is the payoff matrix for player I and B is the payoff matrix for player II, then

$$\max_{x\in\Delta_m}\min_{y\in\Delta_n} x^T A y = \min_{y\in\Delta_n}\max_{x\in\Delta_m} x^T A y.$$
$$\max_{x\in\Delta_m}\min_{y\in\Delta_n} x^T B y = \min_{y\in\Delta_n}\max_{x\in\Delta_m} x^T B y.$$

In words, the first equality says: the maximum over player I's strategies followed by the minimum of the other players strategies of the payoff of player I is equal to the minimum of the other players strategies followed by the maximum over player I's strategies of the payoff of player I.

Now, consider a three-player general-sum game. The analogue of von Neumann's Theorem just applied to player I would say: the maximum over player I's strategies followed by the minimum of the other players strategies of the payoff of player I is equal to the minimum of the other players strategies followed by the maximum over player I's strategies of the payoff of player I.

Show that this statement is false for the following example.

These matrices describe the payoffs for player I. In the game, player I chooses a row (T or B), player II chooses a column (L or R), and player III chooses a matrix (W or E)

	L	R
T	0	1
B	1	1

W

	L	R
T	1	1
B	1	0

E

Exercise 4. Show the following fact, which will be mentioned after our proof of Sperner's Lemma:

Let d be a positive integer. Let K be a closed and bounded subset of $\mathbf{R}^d$. Then the set $K \times K$ is also a closed and bounded set.

(Recall that $K \times K = \{(x, y) \in \mathbf{R}^d \times \mathbf{R}^d \colon x \in K \text{ and } y \in K\} \subseteq \mathbf{R}^{2d}$.)

Exercise 5. Show the following fact, which will be used in the proof of Brouwer's Fixed Point Theorem:

Let d be a positive integer. Let $x^{(1)}, x^{(2)}, \ldots$ be a sequence of points in $\mathbf{R}$. Let $x \in \mathbf{R}$. Assume that $f \colon \mathbf{R} \to \mathbf{R}$ is continuous. Suppose $\lim_{i\to\infty} x^{(i)} = x$. Let $c \in \mathbf{R}$. Assume $f(x^{(i)}) < c$ for all $i \geq 1$. Then $\lim_{i\to\infty} f(x^{(i)}) \leq c$. That is, the limit preserves non-strict inequalities.

(In case you forget the definition of a limit: we say $\lim_{i\to\infty} f(x^{(i)}) = a \in \mathbf{R}$ if, for all $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that, for all $i > N$, we have $\left|f(x^{(i)}) - a\right| < \varepsilon$.)

Exercise 6. Show the following facts, which will be used in our discussion of Correlated equilibria:

For any $x, y \in \Delta_2$, $xy^T \neq \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

For any $x, y \in \Delta_2$, $xy^T \neq \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$.