Please provide complete and well-written solutions to the following exercises.

Due April 26, in the discussion section.

Homework 3

Exercise 1. This exercise deals with subsets of the real line. Show that $[0,1]$ is closed, but $(0,1)$ is not closed.

Exercise 2. This exercise deals with subsets of Euclidean space $\mathbf{R}^d$ where $d \geq 1$. Show that the intersection of two closed sets is a closed set.

Exercise 3. Define $f\colon \mathbf{R}^d \to \mathbf{R}$ by $f(x) := ||x||$. Show that f is continuous. (Hint: you may need to use the triangle inequality, which says that $||x+y|| \leq ||x|| + ||y||$, for any $x, y \in \mathbf{R}^d$. Also, recall that $||(x_1, \ldots, x_d)|| = (\sum_{i=1}^d x_i^2)^{1/2}$.)

Exercise 4. Describe in words the set of points (x_1, x_2) in the plane such that $(x_1, x_2) \geq (3, 4)$.

Exercise 5.

- Let Y be a random variable such that: $Y = 2$ with probability $1/3$, $Y = 3$ with probability $1/3$ and $Y = 5$ with probability $1/3$. What is the expected value of Y?
- Let Z be a random variable such that: $Z = 1$ with probability $1/2$ and $Z = 2$ with probability $1/2$. Assume that Z and Y are independent. What is the probability that: $Y = 3$ and $Z = 2$? What is the expected value of $Y \cdot Z$?

Exercise 6. Let d be a positive integer. Consider

$$\Delta_d := \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d \colon \sum_{i=1}^d x_i = 1,\, x_i \geq 0,\, \forall\, 1 \leq i \leq d\}.$$

Prove that Δ_d is convex, closed and bounded.

Exercise 7.

- Let K be the set of points (x, y) in the plane such that $|x| + |y| \leq 2$. Is K convex? Prove your assertion.
- Let K be the set of points (x, y, z) in $\mathbf{R}^3$ such that $\max(|x|, |y|, |z|) \leq 1/2$. Is K convex? Prove your assertion.
- Let K be the set of points (x, y, z, w) in $\mathbf{R}^4$ such that $x^2 + y^2 + z^2 + w^2 \leq 1$. You may assume that K is convex. Find a hyperplane that separates K from the point $(0, 1, 1, 0)$.

Exercise 8. Show that the intersection of two convex sets is convex. Then, show that the intersection of any finite number of convex sets is convex. Finally, find two convex sets A, B such that the union $A \cup B$ is not convex.

Exercise 9. Let A be an $n \times m$ real matrix. Let $b \in \mathbf{R}^n$, $c \in \mathbf{R}^m$. Using the Minimax Theorem, prove the following equality, which is known as duality for linear programming:

$$\min_{x \in \mathbf{R}^m:\, Ax \geq b,\, x \geq 0} x^T c = \max_{y \in \mathbf{R}^n:\, A^T y \leq c,\, y \geq 0} b^T y$$

(Hint: Consider the game with $(n+m+1) \times (n+m+1)$ payoff matrix given by

$$\begin{pmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{pmatrix}.$$

First, show that the value of the game is 0. Then, apply the Minimax Theorem to this payoff matrix. Using Exercise 10, conclude there exists $x \in \mathbf{R}^m, y \in \mathbf{R}^n, t \in \mathbf{R}$ such that $\sum_{i=1}^m x_i + \sum_{i=1}^n y_i + t = 1$, $x \geq 0$, $y \geq 0$, $t \geq 0$, and such that

$$\begin{pmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \\ t \end{pmatrix} \geq 0.$$

In particular, $b^T y - c^T x \geq 0$. As a simplifying assumption, you may assume $t > 0$. Then, x/t and y/t achieve the minimum and maximum values, respectively, in the duality for linear programming. To show this, prove the following claim. For any $x \in \mathbf{R}^m$ with $Ax \geq b$ and for any $y \in \mathbf{R}^n$ with $A^T y \leq c$, where $x \geq 0, y \geq 0$, we have $c^T x - b^T y \geq 0$.)

Consider now an example where $n = m = 2$, $b = (1, 0)$, $c = (1, 1)$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Using the duality above, show that

$$\max_{y \in \mathbf{R}^n:\, A^T y \leq c,\, y \geq 0} b^T y \leq 1.$$

Exercise 10. Let $x \in \Delta_m$, $y \in \Delta_n$ and let A be an $m \times n$ matrix. Show that

$$\max_{x \in \Delta_m} x^T A y = \max_{i=1,\ldots,m} (Ay)_i, \qquad \min_{y \in \Delta_n} x^T A y = \min_{j=1,\ldots,n} (x^T A)_j.$$

Using this fact, show that

$$\min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y = \min_{y \in \Delta_n} \max_{i=1,\ldots,m} (Ay)_i.$$

$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y = \max_{x \in \Delta_m} \min_{j=1,\ldots,n} (x^T A)_j.$$

Using the second equality, conclude that the value of the game with payoff matrix A can be found via the following Linear Programming problem:

Maximize t subject to the constraints: $\sum_{i=1}^m a_{ij} x_i \geq t$, for all $1 \leq j \leq n$; $\sum_{i=1}^m x_i = 1$; $x \geq (0, \ldots, 0)$.

Efficient methods for solving linear programming problems are well-known. However, below we will focus on ways to compute the values of two-person zero-sum games by hand.