Please provide complete and well-written solutions to the following exercises.

Due March 8th, in the discussion section.

Homework 8

Exercise 1. Prove the following Lemma from the notes: The set of functions $\{W_S\}_{S\subseteq\{1,\ldots,n\}}$ is an orthonormal basis for the space of functions from $\{-1,1\}^n \to \mathbf{R}$, with respect to the inner product defined in the notes. (When we write $S \subseteq \{1,\ldots,n\}$, we include the empty set $\emptyset$ as a subset of $\{1,\ldots,n\}$.) (Also, for any $x \in \{-1,1\}^n$, $W_S(x) = \prod_{i\in S} x_i$.)

Exercise 2. Let $f\colon \{-1,1\}^2 \to \{-1,1\}$ such that $f(x) = 1$ for all $x \in \{-1,1\}^2$. Compute $\widehat{f}(S)$ for all $S \subseteq \{1,2\}$.

Let $f\colon \{-1,1\}^3 \to \{-1,1\}$ such that $f(x_1,x_2,x_3) = \mathrm{sign}(x_1+x_2+x_3)$ for all $(x_1,x_2,x_3) \in \{-1,1\}^3$. Compute $\widehat{f}(S)$ for all $S \subseteq \{1,2,3\}$. The function f is called a **majority function**.

Exercise 3. Let $f\colon \{-1,1\}^3 \to \{-1,1\}$ such that $f(x_1,x_2,x_3) = \mathrm{sign}(x_1+x_2+x_3)$ for all $(x_1,x_2,x_3) \in \{-1,1\}^3$. In the previous homework, we computed $\widehat{f}(S)$ for all $S \subseteq \{1,2,3\}$. The function f is called a **majority function**. Compute the noise stability of f, for any $\rho \in (-1,1)$.

Let n be a positive odd integer. The majority function for n voters can be written as $f(x_1,\ldots,x_n) = \mathrm{sign}(x_1+\cdots+x_n)$, where $x_1,\ldots,x_n \in \{-1,1\}$ and $f\colon \{-1,1\}^n \to \{-1,1\}$. In the limit as $n \to \infty$, the noise stability of the majority function approaches a limiting value. (We implicitly used this fact in stating the Majority is Stablest Theorem.) You will compute this limiting value A in the following way. We have $A = 4B - 1$, where B is defined below.

Let z_1, z_2 be vectors of unit length in $\mathbf{R}^2$. Let $\rho \in (-1,1)$. Let $\cdot$ denote the standard inner product of vectors in $\mathbf{R}^2$. Assume that $z_1 \cdot z_2 = \rho$. Let $C \subseteq \mathbf{R}^2$ be the set such that

$$C = \{(x,y) \in \mathbf{R}^2 \colon (x,y)\cdot z_1 \geq 0 \text{ and } (x,y)\cdot z_2 \geq 0\}.$$

Then

$$B = \iint_C e^{-(x^2+y^2)/2}\,\frac{dxdy}{2\pi}.$$

Compute the value of A. (You should get a relatively simple quantity involving an inverse trigonometric function.)

Exercise 4. Let f denote the majority function for n voters. In class, we showed that $I_i(f) \approx 1/\sqrt{n}$ for all $i \in \{1,\ldots,n\}$. Explain why we can interpret this calculation as saying: your influence in a majority election is a lot more than $1/n$, so you should vote. On the other hand, give reasons why the influence calculation may not accurately reflect your actual influence in a majority election. (If you are thinking of elections in the US, feel free to consider or ignore the electoral college system.)

Exercise 5. Let n be a positive integer. Let $f,g\colon \{-1,1\}^n \to \{-1,1\}$. Let $a_0,\ldots,a_n,b_0,\ldots,b_n \in \mathbf{R}$. Let $x=(x_1,\ldots,x_n)\in\{-1,1\}^n$. For any $x\in\{-1,1\}^n$, define $L_f(x)=a_0+\sum_{i=1}^n a_ix_i$, $L_g(x)=b_0+\sum_{i=1}^n b_ix_i$. Assume that $L_f(x)\neq 0$ and $L_g(x)\neq 0$ for all $x\in\{-1,1\}^n$. Assume also that $f(x)=\mathrm{sign}(L_f(x))$ and $g(x)=\mathrm{sign}(L_g(x))$ for all $x\in\{-1,1\}^n$.

Assume that $\widehat{f}(S)=\widehat{g}(S)$ for all $S\subseteq\{1,\ldots,n\}$ with $|S|\leq 1$. Prove that $f=g$. (Hint: what does the Plancherel Theorem say about $\langle f,L_f\rangle$? How does this quantity compare to $\langle g,L_f\rangle$? Also, note that $f(x)L_f(x)=|L_f(x)|\geq g(x)L_f(x)$ for any $x\in\{-1,1\}^n$.)

Exercise 6. Let n be a positive integer. Show that there is a one-to-one correspondence (or a bijection) between the set of functions f where $f\colon\{-1,1\}^n\to\mathbf{R}$, and the set of functions g where $g\colon 2^{\{1,2,\ldots,n\}}\to\mathbf{R}$. For example, you could identify a subset $S\subseteq\{1,\ldots,n\}$ with the element $x=(x_1,\ldots,x_n)\in\{-1,1\}^n$ where, for all $i\in\{1,\ldots,n\}$, we have $x_i=1$ if $i\in S$, and $x_i=-1$ if $i\notin S$.

Let $i,j\in\{1,\ldots,n\}$ and let $x\in\{-1,1\}^n$. Let $S(x)=\{j\in\{1,\ldots,n\}\colon x_j=1\}$. Using this one-to-one correspondence, show that the i^{th} Shapley value of $f\colon\{-1,1\}^n\to\{-1,1\}$ can be written as

$$\phi_i(f)=\sum_{x\in\{-1,1\}^n\colon x_i=-1}\frac{|S(x)|!(n-|S(x)|-1)!}{n!}\big(f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n)-f(x)\big).$$

So, $\phi_i(f)$ is similar to, but distinct from, $I_i(f)$.

Exercise 7. Let $f\colon\{-1,1\}^n\to\{-1,1\}$. Assume that $\widehat{f}(S)=0$ whenever $S\subseteq\{1,\ldots,n\}$ and $|S|\neq 1$. Show that there exists $i\in\{1,\ldots,n\}$ such that $f(x)=f(x_1,\ldots,x_n)=x_i$ for all $x\in\{-1,1\}^n$, or $f(x)=-x_i$ for all $x\in\{-1,1\}^n$. (This exercise therefore completes the proof of Arrow's Theorem.)