

Please provide complete and well-written solutions to the following exercises.

Due March 1st, in the discussion section.

Homework 7

Exercise 1. Let n be a positive integer. Let $v: 2^{\{1, \dots, n\}} \rightarrow \{0, 1\}$ be a characteristic function that only takes values 0 and 1. Assume also that v is monotonic. That is, if $S, T \subseteq \{1, \dots, n\}$ with $S \subseteq T$, then $v(S) \leq v(T)$. The **Shapley-Shubik power index** of each player is defined to be their Shapley value.

By monotonicity of v , we have $v(S \cup \{i\}) \geq v(S)$ for all $S \subseteq \{1, \dots, n\}$ and for all $i \in \{1, \dots, n\}$. Also, since v only takes values 0 and 1, we have

$$v(S \cup \{i\}) - v(S) = \begin{cases} 1 & \text{when } v(S \cup \{i\}) > v(S) \\ 0 & \text{when } v(S \cup \{i\}) = v(S) \end{cases}.$$

Consequently, we have the following simplified formula for the Shapley-Shubik power index of player $i \in \{1, \dots, n\}$:

$$\phi_i(v) = \sum_{S \subseteq \{1, \dots, n\}: v(S \cup \{i\})=1 \text{ and } v(S)=0} \frac{|S|!(n - |S| - 1)!}{n!}.$$

Compute the Shapley-Shubik power indices for all players on the UN security council, with pre-1965 and post-1965 structure. Which structure is better for nonpermanent members?

In pre-1965 rules, the UN security council had five permanent members, and six nonpermanent members. A resolution passes only if all five permanent members want it to pass, and at least two nonpermanent members want it to pass. So, we can model this voting method, by letting $\{1, 2, \dots, 11\}$ denote the council, and letting $\{1, 2, 3, 4, 5\}$ denote the permanent members. Then we use the characteristic function $v: 2^{\{1, \dots, 11\}} \rightarrow \{0, 1\}$ so that, for any $S \subseteq \{1, \dots, 11\}$, $v(S) = 1$ if $\{1, 2, 3, 4, 5\} \subseteq S$ and if $|S| \geq 7$. And $v(S) = 0$ otherwise.

This voting method was called unfair, so it was restructured in 1965. After the restructuring, the council had the following form (which is still used today). The UN security council has five permanent members, and now ten nonpermanent members. A resolution passes only if all five permanent members want it to pass, and at least four nonpermanent members want it to pass. So, we can model this voting method, by letting $\{1, \dots, 15\}$ denote the council, and letting $\{1, 2, 3, 4, 5\}$ denote the permanent members. Then we use the characteristic function $v: 2^{\{1, \dots, 15\}} \rightarrow \{0, 1\}$ so that, for any $S \subseteq \{1, \dots, 15\}$, $v(S) = 1$ if $\{1, 2, 3, 4, 5\} \subseteq S$ and if $|S| \geq 9$. And $v(S) = 0$ otherwise.

Exercise 2. Let n be a positive integer. Let $v: 2^{\{1, \dots, n\}} \rightarrow \{0, 1\}$ be a characteristic function that only takes values 0 and 1. Assume also that v is monotonic and $v(\{1, \dots, n\}) = 1$. For each $i \in \{1, \dots, n\}$, let B_i be the number of subsets $S \subseteq \{1, \dots, n\}$ such that $v(S) = 0$ and $v(S \cup \{i\}) = 1$. The **Banzhaf power index** of player i is defined to be

$$\frac{B_i}{\sum_{j=1}^n B_j}.$$

Like the Shapley-Shubik power index, the Banzhaf power index is another way to measure the relative power of each player.

Compute the Banzhaf power indices for all players for the glove market example.

Then, compute the Banzhaf power indices for all players on the UN security council, with pre-1965 and post-1965 structure. Which structure is better for nonpermanent members?

Exercise 3. A three-player weighted majority game is given by a set of nonnegative weights $w_1, w_2, w_3 \geq 0$, a cutoff $t \geq 0$, and a characteristic function $v: 2^{\{1, 2, 3\}} \rightarrow \{0, 1\}$ so that, for any $S \subseteq \{1, 2, 3\}$, we have

$$v(S) = \begin{cases} 1 & , \text{ if } \sum_{i \in S} w_i \geq t \\ 0 & , \text{ if } \sum_{i \in S} w_i < t. \end{cases}$$

Compute the Shapley-Shubik power indices and Banzhaf power indices for a general three-player weighted majority game.

Exercise 4. Let V_1, V_2, V_3 be independent random variables that are uniformly distributed in $[0, 1]$. So, for example, for any $0 \leq a < b \leq 1$, the probability that $a \leq V_1 \leq b$ is $b - a$.

Compute the expected value of V_1 . Compute the expected value of $V_1 + V_2 + V_3$. Then compute the expected value of $\max(V_1, V_2, V_3)$. Finally, compute the expected value of V_1^5 .

(For the purposes of this exercise, a random variable X with distribution function $g: \mathbf{R} \rightarrow [0, \infty)$ has expected value $\int_{-\infty}^{\infty} tg(t)dt$. The distribution function g is related to X as follows. For any $a < b$, the probability that $a \leq X \leq b$ is equal to $\int_a^b g(t)dt$. Also, it is assumed that $\int_{-\infty}^{\infty} g(t)dt = 1$.)

Exercise 5. Suppose we have two buyers, and $f(v) = 1$ for any $v \in [0, 1]$ in a sealed-bid second price auction. That is, V_1 and V_2 are uniformly distributed in the interval $[0, 1]$. Show that an equilibrium strategy is $\beta_1(v) = v$, $\beta_2(v) = v$, $\forall v \in [0, 1]$. That is, each player will bid exactly her private value.

Exercise 6 (Muddy Children Puzzle/ Blue-Eyed Islanders Puzzle). This exercise is meant to test our understanding of common knowledge.

Situation 1. There are 100 children playing in the mud. All of the children have muddy foreheads, but any single child cannot tell whether or not her own forehead is muddy. Any child can also see all of the other 99 children. The children do not communicate with each other in any way, there are no mirrors or recording devices, etc. so that no child can see her own forehead. The teacher now says, “stand up if you know your forehead is muddy.” No one stands up, because no one can see her own forehead. The teacher asks again. “Knowing that no one stood up the last time, stand up now if you know your forehead is muddy.” Still

no one stands up. No matter how many times the teacher repeats this statement, no child stands up.

Situation 2. After Situation 1, the teacher now says, “I announce that at least one of you has a muddy forehead.” The teacher then says, “stand up if you know your forehead is muddy.” No one stands up. The teacher pauses then repeats, “stand up if you know your forehead is muddy.” Again, no one stands up. The teacher continues making this statement. The hundredth time that she makes this statement, all the children suddenly stand up.

Explain why all of the children stand up in Situation 2, but they do not stand up in Situation 1. Pay close attention to what is common knowledge in each situation.

Exercise 7. There are five pirates on a ship. It is also common knowledge that every pirate prefers to maximize his amount of gold. There are 100 gold pieces to be split amongst the pirates. The game begins when the first pirate proposes how he thinks the gold should be split amongst the five pirates. All five pirates vote whether or not to accept the proposal, by a majority vote. If the proposal is accepted, the game ends. If the proposal is not accepted, the first pirate is thrown overboard, and the game begins continues. The second pirate now proposes how he thinks the gold should be split amongst the four remaining pirates. All four pirates vote whether or not to accept the proposal, by a majority vote (the current proposer, i.e. the second pirate breaks a tie). If the proposal is accepted, the game ends. If the proposal is not accepted, the second pirate is thrown overboard, and the game continues, etc. (During any voting phase, if a pirate’s share of gold will decrease by throwing the proposer overboard, this pirate will vote to accept the proposal; otherwise this pirate will vote to not accept the proposal.) What is the largest amount of gold that the first pirate can obtain in the game?

Exercise 8. Let X, Y be independent random variables. Let n be a positive integer. Let $p_1, \dots, p_n, q_1, \dots, q_n \geq 0$ with $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$. Let $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbf{R}$. Assume that $X = x_i$ with probability p_i and $Y = y_i$ with probability q_i for all $i \in \{1, \dots, n\}$. Show that the expected value of XY is equal to the expected value of X , multiplied by the expected value of Y .