Please provide complete and well-written solutions to the following exercises.

Due January 12th, in the discussion section.

Homework 1

Exercise 1. Let n be a positive integer. Consider the game Chomp played on an $n \times n$ board. Explicitly describe the winning strategy for the first player. (Hint: the first move should remove the square which is diagonally adjacent to the lower left corner.)

Exercise 2. Compute the following nim-sums: $3 \oplus 4$, $5 \oplus 9$. Then, let a, b, c be nonnegative integers. Prove that $a \oplus a = 0$ and $(a \oplus b) \oplus 0 = a \oplus b$.

Exercise 3. Consider the nim position $(9, 10, 11, 12)$. Which player has a winning strategy from this position, the next player or the previous player? Describe a winning first move.

Exercise 4. Consider the game of Chomp played on a board of size $2 \times \infty$. Recall that a typical Chomp game board is $n \times m$, so that the board has n rows and m columns. We can label the rows as $\{1, 2, \ldots, n\}$ and we can label the columns as $\{1, 2, \ldots, m\}$, where n, m are positive integers. On a $2 \times \infty$ board, we label the rows as $\{1, 2\}$, and we label the columns as $\{1, 2, 3, 4, 5, 6, \ldots\}$. We can think of the row and column labels as coordinates in the xy-plane. So, the lower left corner will still have x-coordinate 1 and y-coordinate 1, so that the lower left square has coordinates $(1, 1)$; the square to the right of this has coordinates $(2, 1)$, and so on.

On the $2 \times \infty$ board, which player has a winning strategy? Prove your assertion, and describe explicitly the winning strategy.

Let $n > 2$ be an integer. On the $n \times \infty$ board, which player has a winning strategy? Prove your assertion, and describe explicitly the winning strategy.

On the $\infty \times \infty$ board, which player has a winning strategy? Prove your assertion, and describe explicitly the winning strategy.

Exercise 5. Let G_1, G_2 be games. Let x_i be a game position for G_i, and let $\mathbf{N}_{G_i}, \mathbf{P}_{G_i}$ denote, $\mathbf{N}$ and $\mathbf{P}$ respectively for the game G_i, for each $i \in \{1, 2\}$. Show the following:

(i) If $x_1 \in \mathbf{P}_{G_1}$ and if $x_2 \in \mathbf{P}_{G_2}$, then $(x_1, x_2) \in \mathbf{P}_{G_1+G_2}$.
(ii) If $x_1 \in \mathbf{P}_{G_1}$ and if $x_2 \in \mathbf{N}_{G_2}$, then $(x_1, x_2) \in \mathbf{N}_{G_1+G_2}$.
(iii) If $x_1 \in \mathbf{N}_{G_1}$ and if $x_2 \in \mathbf{N}_{G_2}$, then (x_1, x_2) could be in either $\mathbf{N}_{G_1+G_2}$ or $\mathbf{P}_{G_1+G_2}$.

Exercise 6. Let G_1, G_2, G_3 be games. Show that the notion of two games being equivalent is an equivalence relation. That is, show the following

- G_1 is equivalent to G_1.
- If G_1 is equivalent to G_2, then G_2 is equivalent to G_1.

- If G_1 is equivalent to G_2, and if G_2 is equivalent to G_3, then G_1 is equivalent to G_3.