167 Final Solutions[1]

1. Question 1

(a) There is a two-person zero sum game with two Nash equilibria and one optimal strategy.

FALSE. In zero sum games, Nash equilibria are equivalent to optimal strategies, i.e. the number of each must be the same. (We showed this on the homework.)

(b) Any correlated equilibrium is a convex combination of Nash equilibria.

FALSE. In the game of chicken, we showed there is a correlated equilibrium which is not a convex combination of Nash equilibria. (We showed this on the homework.)

(c) Every two-person zero sum game has at least one pure Nash equilibrium.

FALSE. The Rock-Paper-Scissors game has only one Nash equilibrium, which is not pure. (We showed this on the homework.)

(d) Every Evolutionarily Stable Strategy (ESS) is a Nash equilibrium.

TRUE. The first condition in the definition of ESS implies that this strategy is a Nash equilibrium. If x is an ESS, then $w^T Ax \leq x^T Ax$ for all pure strategies w. Any mixed strategy y can be expressed as a convex combination of pure strategies $y_1, \ldots, y_n$, i.e. $y = \sum_{i=1}^n t_i y_i$, where $0 \leq t_i \leq 1$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n t_i = 1$. So, $y^T Ax = \sum_{i=1}^n t_i y_i^T Ax \leq \sum_{i=1}^n t_i x^T Ax = x^T Ax$, so that x is a Nash equilibrium.

(e) The Condorcet paradox no longer occurs if we consider an election between four candidates. That is, the Condorcet paradox only occurs in Condorcet elections between three candidates.

FALSE. Consider the exact same example we did in class which demonstrated the Condorcet paradox, which consisted of three voters. Suppose the voters rank candidates a, b, c just as in that example, but they all rank candidate d last. Then society still prefers a over b, b over c, and c over a. So the paradox still occurs.

2. Question 2

State the two-dimensional case of Sperner's Lemma. (Make sure to include all of the assumptions.) (You do **not** have to prove Sperner's Lemma.)

Solution. Suppose we divide a large triangle into smaller triangles, such that the intersection of any two adjacent triangles is a common edge of both. All vertices of the smaller triangles are labelled 1, 2 or 3. The three vertices of the large triangle are labelled 1, 2 and 3. Vertices of small triangles that lie on an edge of the large triangle must receive a label of one of the endpoints of that edge. Given such a labeling, the number of small triangles with three differently labeled vertices is odd; in particular, this number is nonzero.

3. Question 3

Let $A, B \subseteq \mathbb{R}^2$ with $A \cap B = \emptyset$. We say that A, B can be *separated* if the following property holds. There exists $z \in \mathbb{R}^2$ and there exists $c \in \mathbb{R}$ such that $z^T a < c < z^T b$ for all $a \in A$ and for all $b \in B$. We say that A, B cannot be separated if it does not hold that A, B can be separated.

Give an example of two closed, convex sets $A, B \subseteq \mathbb{R}^2$ with $A \cap B = \emptyset$, such that A, B cannot be separated. (As usual, you have to justify your answer. Also, **all** of the required

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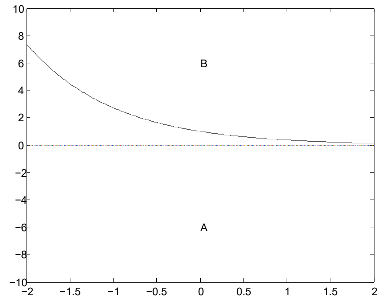

conditions on A, B must be satisfied. Lastly, drawing a picture might be helpful, but it will not constitute a complete answer.)

Solution. There are several examples that work. Here is one example.

Let $A = \{(x, y) \in \mathbb{R}^2 \colon y \leq 0\}$ and let $B = \{(x, y) \in \mathbb{R}^2 \colon y \geq e^{-x}\}$. Then $A \cap B = \emptyset$ since $y > 0$ whenever $(x, y) \in B$, whereas $y < 0$ whenever $(x, y) \in A$. Now, let $z \in \mathbb{R}^2$. Since $z^T a = z^T b$ holds when $z = 0$, assume that $z \neq 0$. If $x = 0$ then $y \neq 0$, and since $(1, 0) \in A$ we have $z^T(1, 0) = 0$, and since $(t, e^{-t}) \in B$ for any $t > 0$, we have $z^T(t, e^{-t}) = ye^{-t}$. Letting $t \to \infty$, then $z^T(t, e^{-t})$ decreases to 0. That is, there does not exist a $c \in \mathbb{R}$ such that $z^T a < c < z^T b$ for all $a \in A$ and for all $b \in B$. Now, if $x \neq 0$, then since $(t, 0) \in A$ for any $t \in \mathbb{R}$, we have $z^T(t, 0) = xt$. So, as t varies over all $t \in \mathbb{R}$, $z^T(t, 0)$ can take any real number value. So, there does not exist $c \in \mathbb{R}$ such that $z^T a < c$ for all $a \in A$. In any case, z does not exist satisfying the condition for A, B being separated.

Lastly, note that A is closed and convex, since it is a closed half plane. Also B is closed since limits preserve nonstrict inequalities (that is, the inequality $y \geq e^{-x}$ is preserved by taking a limit). And B is convex since if $(v, w), (r, u) \in B$, then $w \geq e^{-v}$, $u \geq e^{-r}$, and we are required to show: for any $0 < t < 1$, $(tw + (1 - t)u) \geq e^{-(tv+(1-t)r)}$. To prove this inequality, it then suffices to show that $te^{-v} + (1 - t)e^{-r} \geq e^{-(tv+(1-t)r)}$. Since the function $x \mapsto e^{-x}$ has strictly positive second derivative for any $x \in \mathbb{R}$, Taylor's Theorem implies that, if $b = tv + (1 - t)r$, and if $h(b) = e^{-b}$, $h \colon \mathbb{R} \to \mathbb{R}$, then $h(b + x) \geq h(b) + h'(b)x$. Choosing $x = -tv + tr$ gives $h(r) \geq h(b) + h'(b)t(r - v)$. Choosing $x = -(1 - t)r + (1 - t)v$ gives $h(v) \geq h(b) + h'(b)(1 - t)(v - r)$. Adding these two inequalities, we get the required inequality:

$$te^{-v} + (1 - t)e^{-r} \geq h(b) + h'(b)t(1 - t)(v - r) + h'(b)t(1 - t)(r - v) = h(b).$$

4. Question 4

Prove that any Nash equilibrium is a Correlated Equilibrium. (That is, if m, n are positive integers, and if $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$, then $\widetilde{x}\widetilde{y}^T$ is a correlated equilibrium.) (Here we regard $\widetilde{x}$ and $\widetilde{y}$ as column vectors.)

Solution. We argue by contradiction. Suppose $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium. Let $z = \widetilde{x}\widetilde{y}^T$. Suppose for the sake of contradiction that z is not a correlated equilibrium. Then the negation of the definition of correlated equilibrium holds. Without loss of generality, the negated condition applies to player I. That is, there exists $i, k \in \{1, \ldots, m\}$ such that

$$\sum_{j=1}^{n} z_{ij} a_{ij} < \sum_{j=1}^{n} z_{ij} a_{kj}.$$

That is,

$$\widetilde{x}_i \sum_{j=1}^{n} \widetilde{y}_j a_{ij} < \widetilde{x}_i \sum_{j=1}^{n} \widetilde{y}_j a_{kj}. \qquad (*)$$

This inequality suggests that Player I can benefit by switching from strategy i to strategy k in the mixed strategy $\widetilde{x}$. Let $e_i \in \Delta_m$ denote the vector with a 1 in the i^{th} entry and zeros in all other entries. Define $x \in \Delta_m$ so that $x = \widetilde{x} - \widetilde{x}_i e_i + \widetilde{x}_i e_k$. Observe that

$$x^T A\widetilde{y} - \widetilde{x}^T A\widetilde{y} = (-\widetilde{x}_i e_i + \widetilde{x}_i e_k)^T A\widetilde{y} = -\widetilde{x}_i \sum_{j=1}^{n} a_{ij}\widetilde{y}_j + \widetilde{x}_i \sum_{j=1}^{n} a_{kj}\widetilde{y}_j \overset{(*)}{>} 0.$$

But this inequality contradicts that $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium.

5. Question 5

Define $v\colon 2^{\{1,2,3\}} \to \mathbb{R}$ so that $v(\{1,2\}) = v(\{1,3\}) = 1$, $v(\{2,3\}) = 0$, $v(\{1,2,3\}) = 2$, and $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\emptyset) = 0$.

Using any method you prefer, compute all of the Shapley values of v.

Solution 1. For any $T, S \subseteq \{1,2,3\}$, define $u_T\colon 2^{\{1,2,3\}} \to \mathbb{R}$ such that $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. As shown in class, if $i \in \{1,2,3\}$, then $\phi_i(u_T) = 1/|T|$ if $i \in T$ and $\phi_i(u_T) = 0$ otherwise. (If $T = \emptyset$, then u_T is not a characteristic function, so it's Shapley values are not defined.)

As in the proof of Shapley's Theorem, we want to find coefficients c_T such that

$$v = \sum_{T \subseteq \{1,2,3\}} c_T u_T. \qquad (*)$$

Since $v(\emptyset) = 0$, $(*)$ says $0 = c_\emptyset$ (using $u_T(\emptyset) = 0$ for any $T \neq \emptyset$, $T \subseteq \{1,2,3\}$). If $S \subseteq \{1,2,3\}$ with $|S| = 1$, then $(*)$ says $0 = v(S) = c_S$. So, we can rewrite $(*)$ as $v = \sum_{T \subseteq \{1,2,3\}\colon |T| \geq 2} c_T u_T$. Applying this equality to the set $\{1,2\}$, we have $1 = v(\{1,2\}) = c_{\{1,2\}}$. Similarly, we conclude that $c_{\{1,3\}} = 1$ and $c_{\{2,3\}} = 0$. In summary,

$$v = u_{\{1,2\}} + u_{\{1,3\}} + c_{\{1,2,3\}} u_{\{1,2,3\}}.$$

Applying both sides to the set $\{1,2,3\}$, we get $2 = v(\{1,2,3\}) = 1+1+c_{\{1,2,3\}}$. In conclusion,

$$v = u_{\{1,2\}} + u_{\{1,3\}}.$$

We can now read off the Shapley values of v, using the additivity axiom: $\phi_1(v) = \phi_1(u_{\{1,2\}}) + \phi_1(u_{\{1,3\}}) = 1/2 + 1/2 = 1$. $\phi_2(v) = \phi_2(u_{\{1,2\}}) + \phi_2(u_{\{1,3\}}) = 1/2 + 0 = 1/2$. $\phi_3(v) = \phi_3(u_{\{1,2\}}) + \phi_3(u_{\{1,3\}}) = 0 + 1/2 = 1/2$.

Solution 2. Using a formula from the notes, for any $i \in \{1,2,3\}$, we have

$$\phi_i(v) = \sum_{S \subseteq \{1,2,3\}\colon i \notin S} \frac{|S|!(3 - |S| - 1)!}{3!}(v(S \cup \{i\}) - v(S)).$$

In the case $i = 1$, the only nonzero terms in the sum are $S = \{2\}$, $S = \{3\}$, $S = \{2,3\}$. So,

$$\begin{aligned}\phi_1(v) &= \frac{1}{3!}(v(\{1,2\}) - v(\{2\})) + \frac{1}{3!}(v(\{1,3\}) - v(\{3\})) + \frac{2}{3!}(v(\{1,2,3\}) - v(\{2,3\}))\\ &= \frac{1}{6}(1) + \frac{1}{6}(1) + \frac{1}{3}(2) = 1.\end{aligned}$$

At this point, we could finish by noting that the symmetry axiom implies $\phi_2(v) = \phi_3(v)$, and the efficiency axiom implies $\phi_1(v) + \phi_2(v) + \phi_3(v) = 2$, so that $\phi_2(v) + \phi_3(v) = 2\phi_2(v) = 1$, so that $\phi_2(v) = \phi_3(v) = 1/2$. Alternatively, we could use the above formula again.

In the case $i = 2$, the only nonzero terms in the sum are $S = \{1\}$ and $S = \{1, 3\}$. So,

$$\phi_2(v) = \frac{1}{3!}(v(\{1,2\}) - v(\{2\})) + \frac{2}{3!}(v(\{1,2,3\}) - v(\{1,3\})) = \frac{1}{6}(1) + \frac{1}{3}(1) = 1/2.$$

In the case $i = 3$, the only nonzero terms in the sum are $S = \{1\}$ and $S = \{1, 2\}$. So,

$$\phi_3(v) = \frac{1}{3!}(v(\{1,3\}) - v(\{1\})) + \frac{2}{3!}(v(\{1,2,3\}) - v(\{1,2\})) = \frac{1}{6}(1) + \frac{1}{3}(1) = 1/2.$$

6. Question 6

There are five pirates on a ship. It is also common knowledge that every pirate prefers to maximize his amount of gold. There are 100 gold pieces to be split amongst the pirates. The game begins when the first pirate proposes how he thinks the gold should be split amongst the five pirates. All five pirates vote whether or not to accept the proposal, by a majority vote. If the proposal is accepted, the game ends. If the proposal is not accepted, the first pirate is thrown overboard, and the game begins continues. The second pirate now proposes how he thinks the gold should be split amongst the four remaining pirates. All four pirates vote whether or not to accept the proposal, by a majority vote (the current proposer, i.e. the second pirate breaks a tie). If the proposal is accepted, the game ends. If the proposal is not accepted, the second pirate is thrown overboard, and the game continues, etc. (During any voting phase, if a pirate's share of gold will decrease by throwing the proposer overboard, this pirate will vote to accept the proposal; otherwise this pirate will vote to not accept the proposal.) What is the largest amount of gold that the first pirate can obtain in the game?

Solution. The first pirate can obtain 98 gold pieces and this is the best possible. We argue by working backwards.

If there are two pirates left on the ship, then the fourth pirate can break the tie between the last two pirates. So, the fourth pirate can rationally maximize her gold by claiming all 100 gold pieces.

If there are three pirates left on the ship, the third pirate cannot claim all 100 pieces of gold (the other pirates will reject this proposal, as their payoffs will be larger or equal if they throw the third pirate overboard). However, the third pirate can rationally maximize her gold by claiming 99 pieces of gold for herself by offering 1 piece to the third pirate. This way, the third pirate will accept, since the third pirate's payoff will decrease if he throws overboard the third pirate.

If there are four pirates left on the ship, then the second pirate similarly cannot claim 100 gold pieces, since the other three will vote the second pirate overboard. However, the second pirate can rationally maximize her gold by claiming 99 pieces of gold for herself by offering 1 piece to the fourth pirate. The fourth pirate will accept this offer, since her payoff will decrease by rejecting the offer. And two votes is enough in this case to secure the 99 gold payoff for the second pirate.

Finally, the first pirate cannot claim 100 or 99 gold. But she can rationally maximize her gold by claiming 98 pieces of gold for herself by offering 1 piece to the third pirate, and 1 piece to the fifth pirate. The third and fifth pirate will accept the offer, since not doing so would strictly decrease their payoffs in the next round of voting.

In conclusion, the first pirate can rationally claim 98 gold pieces, and this is the best possible she can do.

7. Question 7

Suppose we have n buyers, and $f(v) = 1$ for any $v \in [0,1]$ in a sealed-bid first price auction. That is, the private values $V_1, \ldots, V_n$ are uniformly distributed in the interval $[0,1]$ (and independent). Show that an equilibrium strategy is $\beta_i(v) = \frac{n-1}{n}v$, for every $v \in [0,1]$, for every $i \in \{1,\ldots,n\}$. (Hint: let $Z = \max(V_2,\ldots,V_n)$. Using probabilistic notation, note that $\mathbf{P}(Z \leq t) = [\mathbf{P}(V_2 \leq t)]^{n-1}$ for all $t \in \mathbb{R}$.)

(In a sealed-bid first price auction, every buyer submits a sealed envelope with her desired bid for the item. The buyer who has submitted the highest bid receives the item for their bid.)

Solution. Let $Z = \max(V_2,\ldots,V_n)$. Let $t \in [0,1]$. Then $Z \leq t$ if and only if $V_i \leq t$ for all $2 \leq i \leq n$. Therefore, $\mathbf{P}(Z \leq t) = \mathbf{P}(V_2 \leq t, \ldots, V_n \leq t)$. By the independence of $V_2,\ldots,V_n$, we get $\mathbf{P}(Z \leq t) = \prod_{i=2}^{n} \mathbf{P}(V_i \leq t) = t^{n-1}$, since $\mathbf{P}(V_i \leq t) = t$ for any $t \in [0,1]$, $2 \leq i \leq n$.

Now, suppose buyers 2 through n all use the equilibrium strategy. And suppose buyer 1 has private value $v \in [0,1]$, and she considers bidding $b \in [0,1]$. Then the expected profit of buyer 1 is $v - b$, multiplied by the probability that she wins the auction. Buyer 1 only wins the auction when $b > \frac{n-1}{n}Z$. So, buyer 1 wins the auction with probability $\mathbf{P}(Z \leq \frac{bn}{n-1}) = \min((\frac{bn}{n-1})^{n-1}, 1)$. That is, the expected profit of buyer 1 is

$$(v-b)\min((\frac{bn}{n-1})^{n-1}, 1).$$

Now, consider the function $b \mapsto (v-b)\min((\frac{bn}{n-1})^{n-1},1)$. If $g(b) = (v-b)(\frac{bn}{n-1})^{n-1}$, then

$$g'(b) = (v-b)n(\frac{bn}{n-1})^{n-2} - (\frac{bn}{n-1})^{n-1} = ((v-b)n - \frac{bn}{n-1})(\frac{bn}{n-1})^{n-2}.$$

So, $g'(b) = 0$ when $(v-b) = \frac{b}{n-1}$, i.e. then $b = \frac{n-1}{n}v$. And this is the unique maximum of the function $b \mapsto (v-b)\min((\frac{bn}{n-1})^{n-1},1)$. So, buyer 1 should bid $b = \frac{n-1}{n}v$, as desired.

8. Question 8

- Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Prove that the noise stability of f is at most 1.
- Let $f\colon \{-1,1\}^n \to \{-1,1\}$ with $\sum_{x\in\{-1,1\}^n} f(x) = 0$. Prove that the noise stability of f with parameter $0 < \rho < 1$ is at most ρ.

Solution. Since $f\colon \{-1,1\}^n \to \{-1,1\}$, then $(f(x))^2 = 1$ for all $x \in \{-1,1\}^n$. So, by Plancherel's Theorem, $1 = \langle f, f\rangle = \sum_{S\subseteq\{1,\ldots,n\}} |\widehat{f}(S)|^2$. Therefore, using $0 < \rho < 1$,

$$\langle f, T_\rho f\rangle = \sum_{S\subseteq\{1,\ldots,n\}} \rho^{|S|}|\widehat{f}(S)|^2 \leq \sum_{S\subseteq\{1,\ldots,n\}} |\widehat{f}(S)|^2 = 1.$$

And equality only occurs when $\widehat{f}(S) = 0$ for all $S \subseteq \{1,\ldots,n\}$ with $|S| \geq 1$. That is, equality only occurs when f is a constant function.

With the additional assumption that $\widehat{f}(\emptyset) = 0$, we similarly have

$$\langle f, T_\rho f\rangle = \sum_{S\subseteq\{1,\ldots,n\}} \rho^{|S|}|\widehat{f}(S)|^2 = \sum_{S\subseteq\{1,\ldots,n\}\colon\, |S|\geq 1} \rho^{|S|}|\widehat{f}(S)|^2 \leq \sum_{S\subseteq\{1,\ldots,n\}\colon\, |S|\geq 1} |\widehat{f}(S)|^2 = 1.$$

And equality only occurs when $\widehat{f}(S) = 0$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| \neq 1$. So, $f\colon \{-1,1\}^n \to \{-1,1\}$ and $\widehat{f}(S) = 0$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| \neq 1$. The following Exercise then completes the proof.

9. Question 9

Let n be a positive integer. Let $f, g\colon \{-1,1\}^n \to \{-1,1\}$. Let $a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbb{R}$. Let $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$. For any $x \in \{-1,1\}^n$, define $L_f(x) = a_0 + \sum_{i=1}^n a_i x_i$, $L_g(x) = b_0 + \sum_{i=1}^n b_i x_i$. Assume that $L_f(x) \neq 0$ and $L_g(x) \neq 0$ for all $x \in \{-1,1\}^n$. Assume also that $f(x) = \mathrm{sign}(L_f(x))$ and $g(x) = \mathrm{sign}(L_g(x))$ for all $x \in \{-1,1\}^n$.

Assume that $\widehat{f}(S) = \widehat{g}(S)$ for all $S \subseteq \{1, \ldots, n\}$ with $|S| \leq 1$. Prove that $f = g$.

Solution. Since $\widehat{L_f}(S) = 0$ whenever $|S| > 1$, Plancherel's Theorem implies that

$$\langle f, L_f \rangle = \sum_{S \subseteq \{1,\ldots,n\}\colon |S| \leq 1} \widehat{f}(S)\widehat{L_f}(S).$$

Also, using our assumptions

$$\begin{aligned}
\langle f, L_f \rangle &= 2^{-n} \sum_{x \in \{-1,1\}^n} f(x) L_f(x) = 2^{-n} \sum_{x \in \{-1,1\}^n} f(x) L_f(x) \\
&= 2^{-n} \sum_{x \in \{-1,1\}^n} f(x)\mathrm{sign}(f(x)) \geq 2^{-n} \sum_{x \in \{-1,1\}^n} g(x)\mathrm{sign}(f(x)) \\
&= \langle g, L_f \rangle = \sum_{S \subseteq \{1,\ldots,n\}\colon |S| \leq 1} \widehat{g}(S)\widehat{L_f}(S) \\
&= \sum_{S \subseteq \{1,\ldots,n\}\colon |S| \leq 1} \widehat{f}(S)\widehat{L_f}(S) = \langle f, L_f \rangle.
\end{aligned}$$

That is, the inequality must be an equality. Since $2^{-n} \sum_{x \in \{-1,1\}^n} f(x)\mathrm{sign}(f(x)) = 2^{-n} \sum_{x \in \{-1,1\}^n} 1 = 1$, we have $2^{-n} \sum_{x \in \{-1,1\}^n} g(x)\mathrm{sign}(f(x)) = 1$. And this equality can only occur when $g(x) = f(x)$ for all $x \in \{-1,1\}^n$

10. Question 10

Explain in detail the statement of Arrow's Impossibility Theorem. (You do not need to prove the Theorem, only state it precisely, state how the Condorcet election works, etc.)

Solution. Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Let $x, y, z \in \{-1,1\}^n$, and assume $(3 - x_i y_i - x_i z_i - y_i z_i)/4 = 1$ for all $i \in \{1, \ldots, n\}$. The votes then describe the preferences of voters between three candidates by a Lemma from the notes. For any such votes x, y, z, assume that a Condorcet winner exists. Then f or $-f$ must be a dictatorship.

Denote the three candidates by a, b, c. We think of f as taking an input of votes x, y, z, and giving an output which is a societal preference. More specifically, given the votes x, y, z, the function f determines the societal preference as follows:

- If $f(x) = 1$, the voters prefer a over b, and if $f(x) = -1$, the voters prefer b over a.
- If $f(y) = 1$, the voters prefer b over c, and if $f(y) = -1$, the voters prefer c over b.
- If $f(z) = 1$, the voters prefer c over a, and if $f(z) = -1$, the voters prefer a over c.

So, for any $1 \leq i \leq n$, x_i is the preference of voter i between candidates a and b, y_i is the preference of voter i between candidates b and c, and z_i is the preference of voter i between candidates c and a.

Given the votes x, y, z, we say that a Condorcet winner of the election exists if one candidate is preferred over the other two by society. From a Lemma from the notes Condorcet winner exists when $(3 - f(x)f(y) - f(x)f(z) - f(y)f(z))/4 = 1$. The usual Condorcet election corresponds to f being the majority function, $f(x_1, \ldots, x_n) = \text{sign}(x_1 + \cdots + x_n)$.

The conclusion of Arrow's Theorem says there exists $i \in \{1, \ldots, n\}$ such that $f(x_1, \ldots, x_n) = x_i$ or $f(x_1, \ldots, x_n) = -x_i$, for all $x \in \{-1, 1\}^n$.