167 Midterm 2 Solutions[1]

1. Question 1

Recall the prisoner's dilemma, which has the following payoffs.

Prisoner I \ Prisoner II	silent	confess
silent	$(-1,-1)$	$(-10,0)$
confess	$(0,-10)$	$(-8,-8)$

Find all Nash equilibria for this game.

Solution. See Example 4.6 in the notes. There it was shown that $x = (0, 1)$ and $y = (0, 1)$ is the only Nash equilibrium for this game.

2. Question 2

Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 3 & 1 & 2 & 0 \\ 4 & 0 & 5 & 3 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$

Solution. The average of the last two rows dominates the first row, so the first row can be ignored in the computation of the value. That is, we can equivalently compute the value of the matrix

$$\begin{pmatrix} 4 & 0 & 5 & 3 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$

The third column dominates the fourth, and the first column dominates the fourth, so the first and third column can be ignored. That is, we have reduced to computing the value of the matrix

$$\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = A.$$

Write $x = (s, 1-s)$, $y = (t, 1-t)$, $s, t \in [0, 1]$. Then using an Exercise from the notes,

$$\max_{x \in \Delta_2} x^T A y = \max_{i=1,2} (Ay)_i = \max(3t, 3(1-t)).$$

And the minimum of this function over all $t \in [0, 1]$ occurs when $3t = 3(1-t)$, i.e. when $6t = 3$, or $t = 1/2$. So, the value of the game is

$$\min_{y \in \Delta_2} \max_{x \in \Delta_2} x^T A y = \min_{t \in [0,1]} \max(3t, 3(1-t)) = 3/2.$$

3. Question 3

Let $f\colon [0,1] \to [0,1]$ be a continuous function. Prove that there exists $x \in [0,1]$ such that $f(x) = x$. (That it, prove the Brouwer Fixed Point Theorem in dimension 1. That is, you cannot just say this follows from the Brouwer Fixed Point Theorem, you have to prove it.) (Hint: you can freely use the Intermediate Value Theorem.)

Solution. Let $g(x) = f(x) - x$. Then g is continuous, since it is the sum of two continuous functions. By assumption $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. So, $f(0) - 0 \geq 0$ and $f(1) - 1 \leq 0$. That is, $g(0) \geq 0$ and $g(1) \leq 0$. So, by the Intermediate Value Theorem, there exists $x \in [0,1]$ such that $g(x) = 0$, i.e. such that $f(x) - x = 0$, i.e. such that $f(x) = x$.

4. Question 4

(i) Give an example of a closed and convex subset K of Euclidean space, and give an example of a continuous function $f\colon K \to K$ such that f has no fixed point. (That is, for every $x \in K$, $f(x) \neq x$.)

Solution. Let $K = \mathbb{R}$ and let $f(x) = x + 1$ for all $x \in \mathbb{R}$. Then K is closed and convex, but f has no fixed points, since the equation $f(x) = x$ says $x + 1 = x$, so that $1 = 0$, i.e. $x \in \mathbb{R}$ does not exist satisfying this equation.

(ii) Give an example of a convex and bounded subset K of Euclidean space, and give an example of a continuous function $f\colon K \to K$ such that f has no fixed point.

Solution. Let $K = (0,1)$ and let $f(x) = x/2$ for all $x \in \mathbb{R}$. Then K is convex and bounded, and f is continuous, but f has no fixed points, since $f(x) = x$ would say $x/2 = x$, i.e. $1/2 = 1$ (since $x > 0$), which can never be satisfied.

(iii) Give an example of a function $f\colon [0,1] \to [0,1]$ such that f has no fixed point.

Solution. For any $x \in [0,1)$, let $f(x) = 1$, and let $f(1) = 0$. Then f has no fixed point, since $f(x) = x$ is never satisfied, by definition of f.

5. Question 5

Prove that any Nash equilibrium is a Correlated Equilibrium. (That is, if m, n are positive integers, and if $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$, then $\widetilde{x}\widetilde{y}^T$ is a correlated equilibrium.) (Here we regard $\widetilde{x}$ and $\widetilde{y}$ as column vectors.)

Solution 1. Suppose $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium. Let $z = \widetilde{x}\widetilde{y}^T$. From a Lemma from the notes, it suffices to show: for any functions $f\colon \{1,\ldots,m\} \to \{1,\ldots,m\}$, $g\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}a_{ij} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}a_{f(i)j}, \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}b_{ij} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}b_{ig(j)}.$$

Define $x \in \Delta_m$ so that, for any $i \in \{1,\ldots,m\}$, $x_i = \sum_{i' \in \{1,\ldots,m\}\colon f(i')=i} \widetilde{x}_{i'}$. Then, using $z_{ij} = \widetilde{x}_i\widetilde{y}_j$, we have $\sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}a_{f(i)j} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij}\widetilde{y}_j$. That is, it suffices to show: for any $x \in \Delta_m, y \in \Delta_n$, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \widetilde{x}_i a_{ij}\widetilde{y}_j \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij}\widetilde{y}_j, \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} \widetilde{x}_i b_{ij}\widetilde{y}_j \geq \sum_{i=1}^{m}\sum_{j=1}^{n} \widetilde{x}_i b_{ij} y_j.$$

That is, it suffices to show: for any $x \in \Delta_m$, for any $y \in \Delta_m$, we have

$$\widetilde{x}^T A\widetilde{y} \geq x^T A\widetilde{y}, \qquad \widetilde{x}^T A\widetilde{y} \geq \widetilde{x}^T Ay.$$

The last condition is exactly the condition defining a Nash equilibrium.

Solution 2. We argue by contradiction. Suppose $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium. Let $z = \widetilde{x}\widetilde{y}^T$. Suppose for the sake of contradiction that z is not a correlated equilibrium. Then the negation of the definition of correlated equilibrium holds. Without loss of generality, the negated condition applies to player I. That is, there exists $i, k \in \{1, \ldots, m\}$ such that

$$\sum_{j=1}^{n} z_{ij}a_{ij} < \sum_{j=1}^{n} z_{ij}a_{kj}.$$

That is,

$$\widetilde{x}_i \sum_{j=1}^{n} \widetilde{y}_j a_{ij} < \widetilde{x}_i \sum_{j=1}^{n} \widetilde{y}_j a_{kj}. \qquad (*)$$

This inequality suggests that Player I can benefit by switching from strategy i to strategy k in the mixed strategy $\widetilde{x}$. Let $e_i \in \Delta_m$ denote the vector with a 1 in the i^{th} entry and zeros in all other entries. Define $x \in \Delta_m$ so that $x = \widetilde{x} - \widetilde{x}_i e_i + \widetilde{x}_i e_k$. Observe that

$$x^T A\widetilde{y} - \widetilde{x}^T A\widetilde{y} = (-\widetilde{x}_i e_i + \widetilde{x}_i e_k)^T A\widetilde{y} = -\widetilde{x}_i \sum_{j=1}^{n} a_{ij}\widetilde{y}_j + \widetilde{x}_i \sum_{j=1}^{n} a_{kj}\widetilde{y}_j \overset{(*)}{>} 0.$$

But this inequality contradicts that $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium.