Please provide complete and well-written solutions to the following exercises.

Due November 29, in the discussion section.

Homework 8

Exercise 1. Let $g\colon [0,1]\to\mathbf{R}$ be a continuous function. Assume that, for any C^1 function $h\colon [0,1]\to\mathbf{R}$ with $h(0)=h(1)=0$, we have

$$\int_0^1 g(x)h(x)dx = 0.$$

Conclude that $g(x)=0$ for all $x\in[0,1]$. (Hint: Argue by contradiction. Assume g is nonzero somewhere. Use the definition of continuity of g to show that g is nonzero in some interval. Then choose h such that h is only nonzero on this interval.)

Exercise 2. Let $D=\{(x,y)\in\mathbf{R}^2\colon x^2+y^2\leq 1\}$ be the unit disc in the plane and let $\partial D=\{(x,y)\in\mathbf{R}^2\colon x^2+y^2=1\}$ be the boundary of D. Let A be the set of all functions $f\colon D\to\mathbf{R}$ such that f is a C^2 function, such that $f(x)=0$ for all $x\in\partial D$, and $f(0)=1$. Show that there does not exist a function $g\in A$ such that

$$\iint_D ||\nabla g(x)||\, dx = \min_{f\in A}\iint_D ||\nabla f(x)||\, dx$$

(Hint: it suffices to find a sequence of functions $f_1,f_2,\ldots\in A$ where $\iint_D ||\nabla f_k(x)||\, dx\to 0$ as $k\to\infty$. Why is this sufficient? Second hint: it may be easier to use polar coordinates.)

Exercise 3 (Isoperimetric Inequality in the Plane)**.** Let $t\in[0,1]$. Let $s(t)=(x(t),y(t))$ be a parametrization of a curve in the plane. Assume that $s(t)\neq s(t')$ for all $t,t'\in[0,1)$ with $t\neq t'$. Assume also that $s(0)=s(1)$. That is, the curve does not intersect itself except at the points $t=0$ and $t=1$.

Assume that x,y are C^1 functions. The length of the curve s is defined to be

$$L(s) := \int_0^1 ||s'(t)||\, dt = \int_0^1 \sqrt{(x'(t))^2+(y'(t))^2}dt.$$

The area enclosed by the curve is defined to be

$$A(s) := \frac{1}{2}\int_0^1 x(t)y'(t)-y(t)x'(t)\, dt.$$

Let $c>0$. Subject to the constraint $L(s)=c$, we wish to maximize $A(s)$.

In order to solve this constrained maximization problem, we assume there exists a curve s of maximal area and of length c. And we also assume that the Lagrange Multiplier Theorem applies to this problem. That is, there exists $\lambda \in \mathbf{R}$ such that, if $r(t) = (w(t), z(t))$ is any function with $t \in [0,1]$, and if $s_p(t) = s(t) + pr(t)$ for any $p \in (-1,1)$, then

$$\frac{d}{dp}L(s_p) = \lambda \frac{d}{dp}A(s_p).$$

By investigating this equation, conclude that s is a circle. That is, there exists $c, d \in \mathbf{R}$ such that $(x-c)^2 + (y-d)^2 = \lambda^2$. Deduce the **isoperimetric inequality**: for any curve s such that $x, y \in C^1$, $s(0) = s(1)$ and such that s does not intersect itself, we have

$$A(s) \leq \frac{1}{4\pi}(L(s))^2.$$

(Hint: first consider r where $w(t) = 0$ for all $t \in [0,1]$, and then consider r where $z(t) = 0$ for all $t \in [0,1]$. Also, you may assume that s is a constant-speed parametrization, so that $||s'(t)||$ is constant for all $t \in [0,1]$.)

Exercise 4. On a previous homework, we investigated the largest entropies on finite-dimensional vector spaces. (The smallest possible entropy is zero, so minimizing entropy is not so interesting.) The infinite-dimensional case is a bit different than the finite-dimensional case.

Let A be the set of all $f\colon \mathbf{R} \to [0,\infty)$ such that $\int_{-\infty}^{\infty} f(x)dx = 1$. (In probability terminology, f is a probability density function.) Maximize the entropy

$$-\int_{-\infty}^{\infty} f(x) \log f(x)\, dx$$

over the set A, subject to the constraint

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx = 1.$$

(You may assume that the maximum exists.) (Hint: if $g \in A$, what functions h can we add to g so that $g + h \in A$?)