Please provide complete and well-written solutions to the following exercises.

Due November 1, in the discussion section.

# Homework 5

**Exercise 1.** Is $\{(x_1, x_2) \in \mathbf{R}^2 \colon x_1^2 + x_2^2 \leq 1\}$ a polytope? Prove your assertion.

**Exercise 2.** Find all extreme points of the set $\{(x_1, x_2) \in \mathbf{R}^2 \colon 0 \leq x_1 \leq 1,\, 0 \leq x_2 \leq 1\}$. Then, find all extreme points of the set $\{(x_1, x_2) \in \mathbf{R}^2 \colon x_1^2 + x_2^2 \leq 1\}$.

**Exercise 3.** Prove that a polytope is convex. Then, draw the following polytopes in the plane:

$$\{(x_1, x_2) \in \mathbf{R}^2 \colon x_1 \geq 0,\, x_2 \geq 0,\, x_1 + x_2 \leq 2,\, x_2 \leq 1\}.$$
$$\{(x_1, x_2) \in \mathbf{R}^2 \colon x_1 \geq 0,\, x_2 \geq 0,\, x_1 + x_2 \leq 1,\, x_1 + 2x_2 = 1\}.$$
$$\{(x_1, x_2) \in \mathbf{R}^2 \colon x_1 \geq 0,\, x_2 \geq 0,\, x_1 + x_2 = 1,\, x_1 + 2x_2 = 1\}.$$

**Exercise 4.** Let $K \subseteq \mathbf{R}^n$ be a polytope formed by the intersection of $m > n$ half spaces. Assume that $K$ is nonempty and bounded. Show that $K$ has at most $\binom{m}{n} = \frac{m!}{(m-n)!n!}$ vertices.

(Hint: first show that a vertex of the polytope must be in the boundary of at least $n$ half spaces, using linear algebra. That is, for any vertex of $K$, there exist at least $n$ of the half spaces that define $K$ such that equality occurs in the definition of that half space.)

**Exercise 5.** Let $K \subseteq \mathbf{R}^n$ be a polytope. Let $f \colon K \to \mathbf{R}$ be a concave function (so that $-f$ is convex). Assume that a minimum value of $f$ exists. That is, there exists $x \in K$ such that $f(x) \leq f(k)$ for all $k \in K$. Conclude that there exists an extreme point $y \in K$ such that $f$ attains its minimum value at $y$.

**Exercise 6.** Let $K$ be a bounded polytope and let $f \colon K \to \mathbf{R}$ be a linear function. Show that the minimum value of $f$ is attained at a vertex of $K$. (Hint: a linear function is concave. Also, from Exercise 4, there are only finitely many vertices of $K$. Using the definition of a vertex, show that: for any point $k \in K$, there exists a vertex $x \in K$ where $f(x) \leq f(k)$.)

**Exercise 7.** Using the Simplex Algorithm, solve the following linear program:

$$\text{minimize} \quad -4x_1 - 2x_2 \quad \text{subject to the constraints}$$

$$x_1 + x_2 + x_3 = 5$$
$$2x_1 + x_2/2 + x_4 = 8, \qquad x \geq 0.$$

(Hint: start at the point $(x_1, x_2, x_3, x_4) = (0, 0, 5, 8)$)

**Exercise 8.** Let $H_n := \{(x_1, \ldots, x_n) \in \mathbf{R}^n \colon 0 \leq x_i \leq 1, \forall\, 1 \leq i \leq n\}$. The set $H_n$ is the $n$-dimensional cube. First, show that $H_n$ is a polytope which is formed by the intersection of $2n$ half spaces. Then, show that $H_n$ has $2^n$ vertices.

**Exercise 9.** Let $C \subseteq \mathbf{R}^n$. Let $A$ be a positive semidefinite matrix. Show that

$$\mathrm{vol}(AC) = \mathrm{vol}\{Ax \in \mathbf{R}^n \colon x \in C\} = \det(A)\mathrm{vol}(C).$$

**Exercise 10.** Let $A$ be a positive definite $n \times n$ matrix. Let $y \in \mathbf{R}^n$ and define the ellipsoid $E(A, y) := \{x \in \mathbf{R}^n \colon (x - y)^T A^{-1}(x - y) \leq 1\}$. Let $z \in \mathbf{R}^n$, $z \neq 0$, and define the half-ellipsoid

$$E'(A, y, z) := E(A, y) \cap \{x \in \mathbf{R}^n \colon \langle z, (x - y)\rangle \leq 0\}.$$

Define

$$d := \frac{1}{\sqrt{z^T A z}} A z.$$

$$y' := y - \frac{1}{n+1} d.$$

$$A' := \frac{n^2}{n^2 - 1}\left(A - \frac{2}{n+1} d d^T\right).$$

Show that $E'(A, y, z) \subseteq E(A', y')$. The set $E(A', y')$ is called the **Löwner-John ellipsoid** of $E'(A, y, z)$. Justify why $A'$ is positive definite?

(For simplicity, you may assume that $A$ is the identity matrix, $y = 0$ and $z = (1, 0, \ldots, 0)$. The case of general $A, y, z$ is left as an optional challenge problem. For the general problem, the Sherman-Morrison formula may be helpful.)

**Exercise 11.** By taking logarithms, show that for any positive integer $n$,

$$\left(\frac{n}{n+1}\right)^{n+1} \left(\frac{n}{n-1}\right)^{n-1} \leq e^{-1/n}.$$

(Hint:)

$$\begin{aligned}
&(n+1)\log(1 + 1/n) + (n-1)\log(1 - 1/n) \\
&= \sum_{k=1}^{\infty} (-1)^{k+1}(n+1)n^{-k}k^{-1} - \sum_{k=1}^{\infty} (n-1)n^{-k}k^{-1} \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} 2n^{-k}k^{-1} + \sum_{k=1}^{\infty} (-1)^{k+1}(n-1)n^{-k}k^{-1} - \sum_{k=1}^{\infty} (n-1)n^{-k}k^{-1} = \cdots
\end{aligned}$$