118 Spring 2018 Final Solutions[1]

1. Question 1

You sell handmade, made-in-the-USA T-shirts. The average cost function for your operation is given by $AC(q) = \frac{1200}{q} + 500 + 0.01(q100)^2$,

where q is the number of T-shirts produced and $AC(q)$ is measured in cents per shirt.

(a) Find a formula for your total costs.

Solution. $C = qAC(q) = 1200 + 500q + .01(q-100)^2 q$.

(b)) What are your fixed costs?

Solution. 1200. This is the constant term appearing in C (the part that does not depend on q).

(c) Find a formula for your marginal cost.

Solution. $MC(q) = \frac{d}{dq}C = 500 + .01(q-100)(2)q + .01(q-100)^2$.

(d) What is the minimum of marginal cost in the interval $100 \leq q \leq 400$?

Solution. We take the derivative and look for critical points. We have

$$MC'(q) = .02(q-100) + .02q + .02(q-100) = .06q - 4 = 0.$$

Solving for q, we get $q = 4/(6/100) = 400/6. = 200/3 \approx 67$. So, MC has no critical points in the interval $[100, 400]$. So, the minimum of MC must occur at the endpoints of this interval. We have $MC(100) = 500$ and $MC(400) = 500 + .01(300)(2)(400) + .01(300)^2 = 500 + 2400 + 900 = 3800$. So, the minimum occurs at $q = 100$, with minimum marginal cost 500.

2. Question 2

Suppose that $g(x) = 13x$, and $f(x)$ is represented by the graph:

(a) Find $\frac{d}{dx}[f(x)g(x)]$ at $x = 5$.

Solution. From the product rule, $\frac{d}{dx}[f(x)g(x)]|_{x=5} = f'(5)g(5) + g'(5)f(5) = f'(5)(14) + (-3)f(5)$. From the picture, $f'(5) = (-1-3)/2 = -2$ and $f(5) = 1$. So $\frac{d}{dx}[f(x)g(x)]|_{x=5} = -2(14) + (-3)(1) = -31$.

(b) Find $\frac{d}{dx}[f(x)/g(x)]$ at $x = 2$.

From the quotient rule, $\frac{d}{dx}[f(x)/g(x)]|_{x=2} = \frac{g(2)f'(2)-f(2)g'(2)}{(g(2))^2} = \frac{(-5)f'(2)-f(2)(-3)}{(-5)^2} = \frac{-5f'(2)+3f(2)}{25}$. From the picture, $f'(2) = (1-0)/2 = 1/2$ and $f(2) = 1/2$. So $\frac{d}{dx}[f(x)/g(x)]|_{x=2} = \frac{-5(1/2)+3(1/2)}{25} = \frac{-2}{50} = -\frac{1}{25}$.

3. Question 3

A hockey team plays in an arena with a seating capacity of 15, 000. At \$12 per ticket, they have an average attendance of 11, 000 fans. A recent survey shows that for every dollar they decrease their ticket price, their attendance will increase by 1, 000 fans. Where should they set their price for maximum revenue?

Solution. Let p be the price of the ticket in dollars. It is given that fan attendance is $11000 + (p-12)1000$, so revenue R satisfies $R = p(11000 + (p-12)1000) = 1000p^2 - 1000p$. Then $R'(p) = 1000(2p-1)$ so $R'(1/2)$=0. And $R'(p) < 0$ when $p < 1/2$ and $R'(p) > 0$ when $p > 1/2$, so the absolute minimum of R occurs when $p = 1/2$. To find the absolute maximum,

[1]October 6, 2018,

we have to check the endpoints of the interval. A negative ticket price is unrealistic so we must have $p \geq 0$. And the arena has a maximum capacity of 15000, so we must have $p \leq 16$. In summary, $0 \leq p \leq 16$. Since $R(0) = 0$, the absolute maximum revenue occurs when $p = 16$, so this should be the ticket price.

4. Question 4

(20 points). You make beer. Let $C(q)$ denote the cost in dollars of producing q pints, $MC(q)$ denote the marginal cost, and $AC(q)$ the average cost.

(a) Suppose $\int_{40}^{70} MC(q)dq = 20$. What does this tell us? Choose the best option.

- The marginal cost, when you have produced 40 pints of beer, is \$20 per pint.
- Suppose you have already made 40 pints of beer. The next 30 pints will cost, on average, \$20 per pint.
- Producing 20 pints of beer will cost $70 - 40 = 30$ dollars.
- Suppose you have already made 40 pints of beer. It will cost \$20 to produce the next 30 pints. [X] [By the fundamental theorem of calculus, the equality becomes $C(70) - C(40) = 20$.]

(b) Suppose $\frac{1}{35} \int_{100}^{135} MC(q)dq = 1/2$. What does this tell us? Choose the best option.

- The marginal cost, when you have produced 100 pints of beer, is \$0.50 per pint.
- Suppose you have already made 100 pints of beer. The next 35 pints will cost, on average, \$0.50 per pint. [X] [By the fundamental theorem of calculus, the equality becomes $\frac{C(135)-C(100)}{35} = 1/2$. The left quantity is the average cost of production from 100 to 135 pints.]
- Producing 1/2 a pint of beer will cost 35 dollars
- Suppose you have already made 100 pints of beer. It will cost \$0.50 to produce the next 35 pints.

(c) Consider the value of q where $MC(q) = AC(q)$. Which of the following are true about this value of q?

- This is the quantity where the profit is maximized.
- This is the quantity where you break even.
- This is a critical point of the average cost function. [X] [Since $AC(q) = C(q)/q$, we have $AC'(q) = [qC'(q) - C(q)]/q^2$. So, $AC'(q) = 0$ exactly when $qC'(q) - C(q) = 0$, i.e. when $C'(q) = C(q)/q = AC(q)$, i.e. $MC(q) = AC(q)$.]
- This is a critical point of the marginal cost function.

5. Question 5

Given that $f(x, y) = \ln(x\sqrt{x^2 + y^2})$, compute the derivatives:

[For simplicity, we use properties of the natural log function to write $f(x, y) = \ln x + (1/2)\ln(x^2 + y^2)$.]

(a) $\frac{\partial}{\partial x}$

Solution. From the Chain Rule,

$$\frac{\partial}{\partial x} = \frac{1}{x} + (1/2)\frac{2x}{x^2 + y^2} = \frac{1}{x} + \frac{x}{x^2 + y^2}.$$

(b) f_y

Solution. From the Chain Rule,

$$f_y = (1/2)\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}.$$

(c) $\frac{\partial^2}{\partial x \partial y}$

Solution. Using part (b) and the quotient rule, we have

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x} f_y = \frac{(x^2+y^2)(0) - y(2y)}{(x^2+y^2)^2} = \frac{-2y^2}{(x^2+y^2)^2}.$$

6. Question 6

Compute the following integrals.

(a) $\int 2^{x-1}dx$.

Solution. Since $(d/dx)2^x = 2^x \ln(2)$, we have $2^{x-1} = (\ln 2)^{-1}(d/dx)2^x$, so that

$$\int 2^{x-1}dx = (\ln 2)^{-1}2^x + C.$$

(b) $\int_2^3 2te^{t^2}dt$.

Solution. We use u-substitution of the form $u = t^2$ so that $du = 2tdt$ and

$$\int_2^3 2te^{t^2}dt = \int_4^9 e^u du = [e^u]_{u=4}^{u=9} = e^9 - e^4.$$

(c) $\int (\ln y)^2 dy$.

Solution. We integrate by parts using $\int udv = uv - \int vdu$ where $u = (\ln y)^2$ and $v = y$. Then $dv = dy$ and $du = 2\ln y(1/y)dy$, so

$$\int (\ln y)^2 dy = y(\ln y)^2 - \int 2y\ln(y)(1/y)dy = y\ln y - 2\int \ln(y)dy.$$

We integrate by parts again using $\int udv = uv - \int vdu$ where $u = \ln y$ and $v = y$. Then $dv = dy$ and $du = (1/y)dy$, so

$$\int (\ln y)^2 dy = y(\ln y)^2 - 2[y\ln y - \int y(1/y)dy] = y(\ln y)^2 - 2[y\ln y - \int dy] = y(\ln y)^2 - 2y\ln y + 2y + C.$$

7. Question 7

Albert is currently in Kindergarten, and his parents decide to set up a bank account for his college education that will start in 12 years. Assume the annual interest rate is 3%, compounded continuously, and they will deposit money continuously throughout the 12-year period

(a) If they deposit the money at a constant rate of \$9,000 per year, how much money will be in the account at the end of the 12-year period?

Solution. The money in the account is

$$\int_0^{12} e^{.03t}9000dt = 9000\int_0^{12} e^{.03t}dt = 9000\frac{1}{.03}[e^{.03t}]_{t=0}^{t=12} = 9000\frac{100}{3}[e^{.36} - 1] \approx 130,000.$$

(b) If they expect that Albert will need \$200,000 for college education, at what rate should they deposit money?

If they deposit at a constant rate R, then R should satisfy

$$\int_0^{12} e^{.03t} R dt = 200000.$$

Using the above integral computation, we have

$$R\frac{100}{3}[e^{.36} - 1] = 200000, \qquad R = \frac{2000}{3(e^{.36} - 1)} \approx 15385.$$

8. Question 8

Find the critical points of the following function:

$$f(x, y) = 2x^2 + y^2 + 8xy6y + 20$$

Use the second derivative test to determine if your critical points represent local maximums, local minimums, or saddle points.

Solution. The critical points must occur when both partial derivatives are zero. That is, we solve

$$\frac{\partial}{\partial x} f(x, y) = 0, \qquad \frac{\partial}{\partial y} f(x, y) = 0.$$

That is,

$$4x + 8y = 0, \qquad 52y + 8x - 6 = 0.$$

That is, $x = -2y$ and $y = 3 - 4x$, so that $y = 3 - 4(-2y)$, so $y = 3 + 8y$, so $7y = -3$, and $y = -3/7$. Then $x = -2y = 6/7$. So, the only critical point is

$$(x, y) = (6/7, -3/7).$$

We have at this point,

$$\frac{\partial^2}{\partial x^2} f(x, y) = 4, \qquad \frac{\partial^2}{\partial y^2} f(x, y) = 2, \qquad \frac{\partial^2}{\partial x \partial y} f(x, y) = 8.$$

So at the point $(6/7, -3/7)$,

$$\frac{\partial^2}{\partial x^2} f(x, y) \frac{\partial^2}{\partial y^2} f(x, y) - \big(\frac{\partial^2}{\partial x \partial y} f(x, y)\big)^2 = 4(2) - 8^2 < 0.$$

The second derivative test says we must then have a saddle point.

9. Question 9

The following contour diagram shows the corn production C (in millions of kg) in a given year, based on that years average rainfall R (in inches) and average temperature T (in degrees Fahrenheit):

(a) Estimate the values of $\frac{\partial C}{\partial R}$ and $\frac{\partial C}{\partial T}$ at the point $(12, 65)$ marked P, including units.

Solution. [Note: there is more than one way to correctly answer this part.] At the point P, we have the approximation

$$\frac{\partial C}{\partial R} \approx \frac{C(14, 65) - C(12, 65)}{14 - 12} = \frac{80 - 70}{2} = 5.$$

$$\frac{\partial C}{\partial T} \approx \frac{C(12, 70) - C(12, 65)}{70 - 65} = \frac{60 - 70}{5} = -2.$$

(b) Use the derivatives in Part (a) to estimate the corn production C at $(11.5, 65.8)$

Solution. Using linear approximation, we have

$$C(12+a,65+b)\approx C(12,65)+a\frac{\partial C}{\partial R}(12,65)+b\frac{\partial C}{\partial T}(12,65).$$

So, using part (a),

$$C(11.5,65.8)=C(12+(-.5),65+.8)\approx C(12,65)+(-.5)(5)+(.8)(-2)=70-2.5-1.6=70-4.1=65.9.$$

(c) At the point $(18,65)$ marked Q, is $\partial^2C/\partial R^2$ positive or negative? Explain how you know.

Solution. [Note: there is more than one way to correctly answer this part.] Let $f(R)=C(R,65)$. That is, f is the function C restricted to the line where $T=65$ is constant. From the picture, it looks like the point Q is near a local maximum of f. So, we should have $f''(R)<0$. That is $f''(R)=\partial^2C/\partial R^2<0$ at the point Q.

10. Question 10

Consider the function $f(x,y)=x^2+y$, and the triangle T in the x,y-plane with its vertices at $(1,1),(1,2)$, and $(2,2)$.

Set up the integral $\iint_T f(x,y)dA$ in two ways, with the appropriate bounds:

(a) ... as an iterated integral in the order dx dy,

Solution. The hypotenuse of the triangle is the line $y=x$. So, first integrating in x means x goes from -1 to $x=y$, and then y goes from -1 to 2. That is,

$$\iint_T f(x,y)dA=\int_{y=-1}^{y=2}\int_{x=-1}^{x=y}f(x,y)dxdy.$$

(b) ... as an iterated integral in the order dy dx

Solution. The hypotenuse of the triangle is the line $y=x$. So, first integrating in y means y goes from x to 2, and then x goes from -1 to 2. That is,

$$\iint_T f(x,y)dA=\int_{x=-1}^{x=2}\int_{y=x}^{y=2}f(x,y)dydx.$$

(c) Compute $\iint_T f(x,y)dA$ using the ordering that you prefer.

Solution. [I will show both computations.]

$$\begin{aligned}\iint_T f(x,y)dA&=\int_{y=-1}^{y=2}\int_{x=-1}^{x=y}(x^2+y)dxdy=\int_{y=-1}^{y=2}[(1/3)x^3+yx]_{x=-1}^{x=y}dy\\&=\int_{y=-1}^{y=2}[(1/3)y^3+y^2-(1/3)(-1)^3-(-y)]dy\\&=\int_{y=-1}^{y=2}[(1/3)y^3+y^2+(1/3)+y]dy=[(1/12)y^4+(1/3)y^3+(1/3)y+(1/2)y^2]_{y=-1}^{y=2}\\&=(16/12+8/3+2/3+2-1/12-(1/3)(-1)-(1/3)(-1)-(1/2))\\&=(4/3+8/3+2/3+2-1/12+2/3-1/2)=(22/3-1/12-1/2)\\&=(88-1-6)/12=81/12.\end{aligned}$$

$$
\begin{aligned}
\iint_T f(x,y)dA &= \int_{x=-1}^{x=2}\int_{y=x}^{y=2}(x^2+y)dydx \int_{x=-1}^{x=2}[x^2y+y^2/2]_{y=x}^{y=2}dx \\
&= \int_{x=-1}^{x=2}[2x^2+2-x^3-x^2/2]dx = [(2/3)x^3+2x-(1/4)x^4-(1/6)x^3]_{x=-1}^{x=2} \\
&= [16/3+4-4-8/6-(2/3)(-1)^3-2(-1)+(1/4)+(1/6)(-1)^3] \\
&= 16/3-4/3+2/3+2+(1/4)-1/6 = 14/3+2+1/4-1/6 \\
&= (56+24+3-2)/12 = 81/12.
\end{aligned}
$$

(d) Find the average value of $f(x,y)$ over the triangle T.

The average value is the integral of f over T, divided by the area of the triangle. The area of the triangle is $(3)(3)/2 = 9/2$. So, using part (c), the average value is

$$\frac{1}{9/2}\frac{81}{12} = \frac{2}{9}\frac{81}{12} = \frac{9}{6} = \frac{3}{2}.$$