MATH 118 QUIZ SOLUTIONS

STEVEN HEILMAN

Contents

1. Quiz 1

Exercise 1.1. Find the equation for the line passing through the points $(-1, 4)$ and $(2, 6)$.

Solution. We solve for m, b in the equation $y = mx+b$. We have $4 = -m+b$ and $6 = 2m+b$. So, $b = 4 + m$, and $6 = 2m + 4 + m$, so $3m = 2$, and $m = 2/3$. Since $6 = 2m + b$, we have $6 = 4/3 + b$, so $b = 14/3$. In conclusion, $y = (2/3)x + 14/3$. □

Exercise 1.2. Sketch the function $y = x^4$. Then sketch the function $y = x^5$.

Exercise 1.3. Find the equation for the line passing through $(1, 2)$ with slope 3.

Solution. We solve for b in the equation $y = 3x + b$. We have $2 = 3 + b$, so that $b = -1$. That is, $y = 3x - 1$. □

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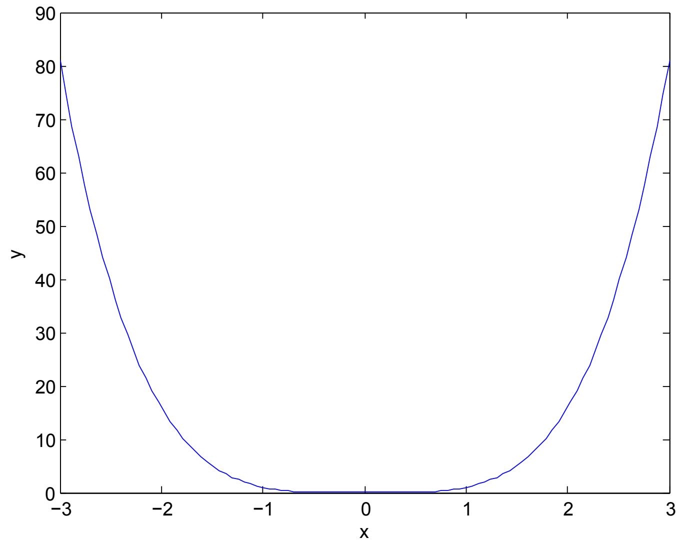
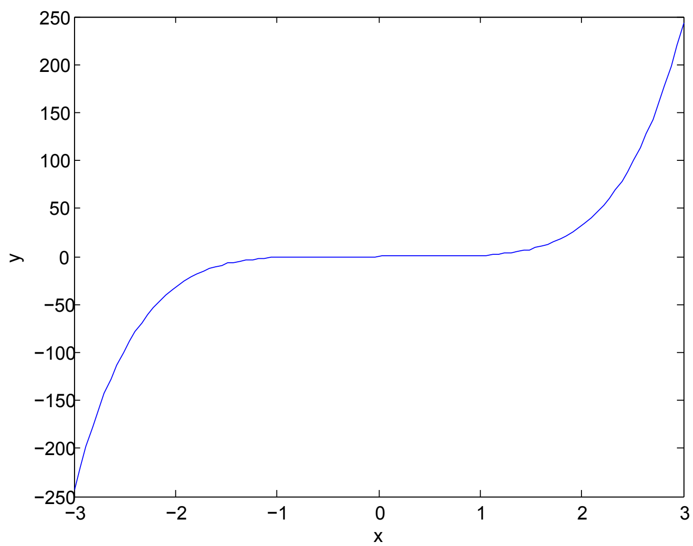




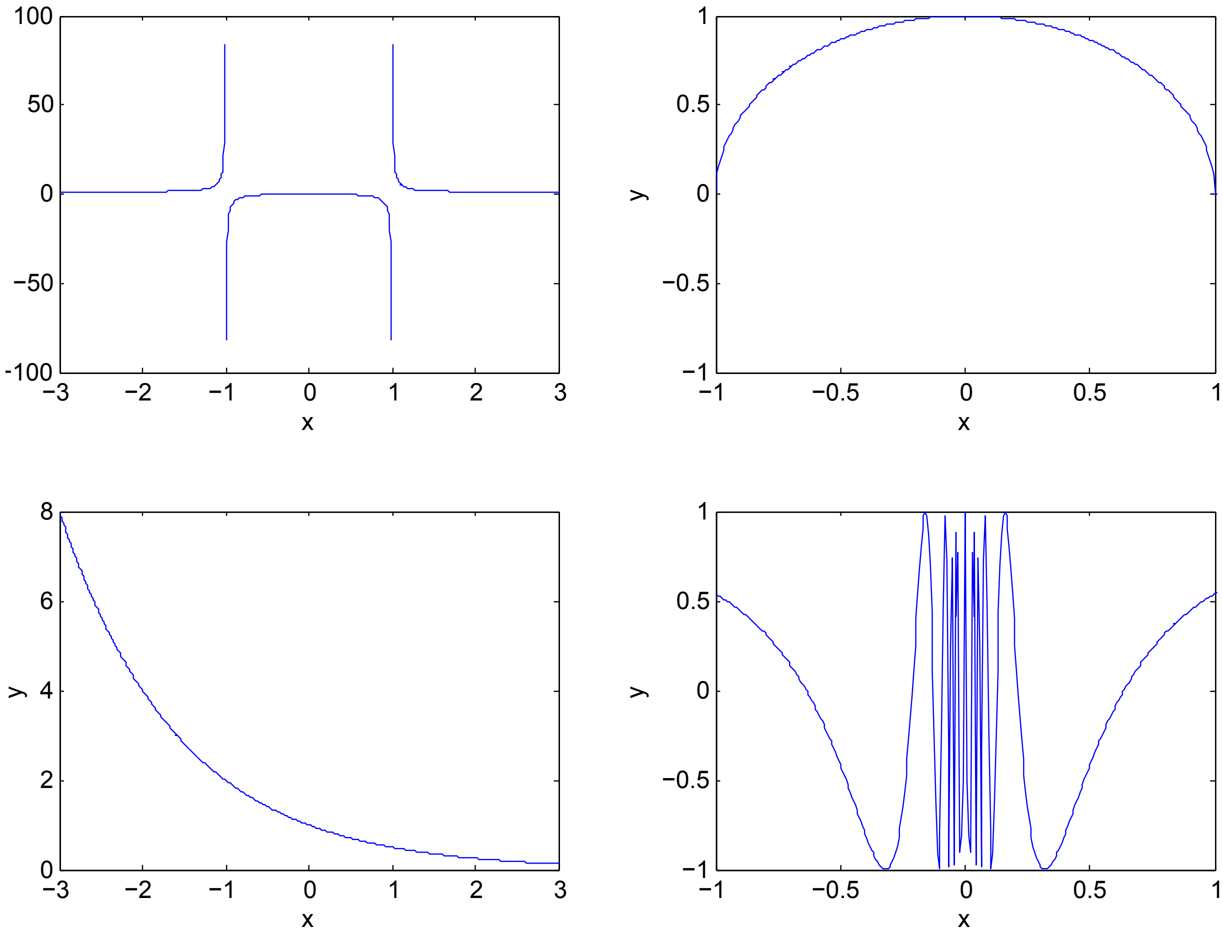
Exercise 1.4.

- Sketch the function $y = \frac{x^2}{x^2-1}$. Is this function even, odd, or neither?
- Sketch the function $y = \sqrt{1-x^2}$. Is this function even, odd, or neither?
- Sketch the function $y = 2^{-x}$. Is this function even, odd, or neither?

Solution. The function $y = \frac{x^2}{x^2-1}$ is even, since $y(-x) = y(x)$.

The function $y = \sqrt{1-x^2}$ is even, since $y(-x) = y(x)$.

The function $y = 2^{-x}$ is neither even nor odd, since $y(-1) = 2 \neq y(1) = 1/2$, and $y(-1) = 2 \neq -y(1) - 1/2$.



$\square$

Exercise 1.5. True or False: For any real number x, we have $\sqrt{x^2} = x$. Justify your answer.

Solution. False. $\sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1$. $\square$

Exercise 1.6. True or False: For any real numbers x, y, we have $|x + y| \leq |x| + |y|$.

Solution. True. If $x > 0$ and $y > 0$, then this is an equality. Similarly, if $x < 0$ and $y < 0$, then this is an equality. If $x > 0$ and $y < 0$, then this inequality holds, since $|x + y|$ is less than the maximum of $|x|$ and $|y|$, so $|x + y| \leq \max(|x|, |y|) \leq |x| + |y|$. Similarly, if $x < 0$ and $y > 0$ then this inequality holds. $\square$

Exercise 1.7. Sketch the region in the plane that solves the inequality

$$|x| + |y| \leq 1.$$

Solution. We will show that this region is a diamond with corners $(1, 0), (-1, 0), (0, 1), (0, -1)$.

We break into four cases that together encompass all $(x,y) \in \mathbf{R} \times \mathbf{R}$.

Case 1. $x \geq 0$ and $y \geq 0$. In this case, $|x| = x$ and $|y| = y$, so we need to find $(x,y) \in \mathbf{R}\times\mathbf{R}$ such that $x + y \leq 1$. In the first quadrant, this is the region under (and including) the line $y = -x + 1$.

Case 2. $x \leq 0$ and $y \geq 0$. In this case, $|x| = -x$ and $|y| = y$, so we need to find $(x,y) \in \mathbf{R}\times\mathbf{R}$ such that $-x + y \leq 1$. In the second quadrant, this is the region under (and including) the line $y = 1 + x$.

Case 3. $x \leq 0$ and $y \leq 0$. In this case, $|x| = -x$ and $|y| = -y$, so we need to find $(x,y) \in \mathbf{R} \times \mathbf{R}$ such that $-x - y \leq 1$. In the third quadrant, this is the region above (and including) the line $y = -x - 1$.

Case 4. $x \geq 0$ and $y \leq 0$. In this case, $|x| = x$ and $|y| = -y$, so we need to find $(x,y) \in \mathbf{R}\times\mathbf{R}$ such that $x - y \leq 1$. In the fourth quadrant, this is the region above (and including) the line $y = x - 1$. $\square$

Exercise 1.8. Consider the curve that satisfies the following equation for $x, y \in \mathbf{R}$

$$x^4 - 4x^2 - x^2y^2 + 4x^2 = 0.$$

Is the curve the graph of a function?

Solution. This curve is not the graph of a function

If this curve were the graph of a function $y(x)$, we would have $x^4 - x^2(y(x))^2 = 0$, i.e. for $x \neq 0$ we would have $(y(x))^2 = x^2$. But then $y(x)$ could have the value x or $-x$. Since there are two possible values for $y(x)$, we conclude that $y(x)$ cannot be a single function, i.e. the given curve is not the graph of a function. $\square$

Exercise 1.9. Solve for x: $x^2 + 5x - 7 = 0$.

Solution. We have $x = (-5 \pm \sqrt{25 + 28})/2$, using the quadratic formula. $\square$

Exercise 1.10. Compute: 2^{2+3}, $(2^2)^3$.

Solution. $2^{2+3} = 2^5 = 32$ and $(2^2)^3 = 2^6 = 64$. $\square$

2. Quiz 2

Exercise 2.1. Define

$$H(x) = \begin{cases} 0 & \text{, if } x < 0 \\ 1 & \text{, if } x \geq 0 \end{cases}.$$

Explain in your own words why $\lim_{x\to 0} H(x)$ does not exist.

Solution. When x is very close to 0 with x negative, we have $H(x) = 0$. However, when x is very close to 0 with x positive, we have $H(x) = 1$. So, no matter how close to zero we look, we can always find nearby x, y with $H(x) = 0$ and $H(y) = 1$. So, it is impossible for $\lim_{x\to 0} H(x)$ to exist, since $H(x)$ is not getting close to any particular value. $\square$

Exercise 2.2. Find two functions f and g so that neither function has a limit as $x \to a$, but $\lim_{x\to a}(f + g)$ exists.

Solution. Let $f(x) = H(x)$ and let $g(x) = -H(x)$. As in the previous Exercise, we know that $\lim_{x\to 0} f(x)$ DNE and $\lim_{x\to 0} g(x)$ DNE. However, $f(x)+g(x) = 0$, so $\lim_{x\to 0}(f(x)+g(x)) = 0$.

Alternately, let $f(x) = 1/x$ and let $g(x) = -1/x$. Then $\lim_{x\to 0} f(x)$ DNE and $\lim_{x\to 0} g(x)$ DNE. However, $f(x) + g(x) = 0$ for $x \neq 0$, so $\lim_{x\to 0}(f(x) + g(x)) = 0$. $\square$

Exercise 2.3. Find all values of a such that

$$\lim_{x\to 0} \frac{\sqrt{ax + 4} - 2}{x} = 1.$$

Solution. We prove that $a = 4$. Let $a \in \mathbf{R}$. For $|x| < 1/|a|$, we have $|ax| < 1$, so the triangle inequality says $|ax + 4| \geq 4 - |ax| \geq 3 > 0$. In particular, $\sqrt{(ax + 4)^2} = |ax + 4| = ax + 4$. For x with $|x| < 1/|a|$, we write

$$\begin{aligned} \frac{\sqrt{ax + 4} - 2}{x} &= \frac{(\sqrt{ax + 4} - 2)}{x} \frac{\sqrt{ax + 4} + 2}{\sqrt{ax + 4} + 2} = \frac{|ax + 4| - 4}{x(\sqrt{ax + 4} + 2)} \\ &= \frac{ax}{x(\sqrt{ax + 4} + 2)} = \frac{a}{\sqrt{ax + 4} + 2}. \end{aligned}$$

So, letting $x \to 0$, we get

$$\lim_{x\to 0} \frac{\sqrt{ax + 4} - 2}{x} = \lim_{x\to 0} \frac{a}{\sqrt{ax + 4} + 2} = \frac{a}{\sqrt{4} + 2} = \frac{a}{4}.$$

In conclusion, we must have $a = 4$. $\square$

Exercise 2.4. Evaluate the following limit and justify each step by indicating the appropriate limit law.

$$\lim_{u\to -2} \sqrt{u^4 + 3u + 6}$$

Solution. Since the square root function is continuous on its domain, we have

$$\begin{aligned} \lim_{u\to -2} \sqrt{u^4 + 3u + 6} &= \sqrt{\lim_{u\to -2}(u^4 + 3u + 6)} \\ &= \sqrt{(\lim_{u\to -2} u^4) + (\lim_{u\to -2} 3u) + (\lim_{u\to -2} 6)} \quad \text{, by Limit Law } (i) \\ &= \sqrt{(\lim_{u\to -2} u)^4 + 3(\lim_{u\to -2} u) + 6} \quad \text{, by Limit Laws } (iii) \text{ and } (ii) \\ &= \sqrt{16 - 6 + 6} = \sqrt{16} = 4. \end{aligned}$$

$\square$

Exercise 2.5. Evaluate the following limit, if it exists. If it does not exist, explain why it does not exist.

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$$

Solution. If $t\neq 0$, we have

$$\frac{1}{t}-\frac{1}{t(t+1)}=\frac{t+1-1}{t(t+1)}=\frac{t}{t(t+1)}=\frac{1}{t+1}.$$

So, the limit exists and is equal to 1:

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\frac{1}{t+1}=\frac{1}{\lim_{t\to 0}(t+1)}=\frac{1}{1}=1.$$

$\square$

Exercise 2.6. Evaluate the following limit, if it exists. If it does not exist, explain why it does not exist.

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}$$

Solution. If $x\neq 0$, we have

$$\frac{x}{\sqrt{1+3x}-1}=\frac{x}{\sqrt{1+3x}-1}\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}=\frac{x(\sqrt{1+3x}+1)}{1+3x-1}=\frac{x(\sqrt{1+3x}+1)}{3x}=\sqrt{1+3x}.$$

So, the limit exists and is equal to 1:

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}=\lim_{x\to 0}\sqrt{1+3x}=\sqrt{\lim_{x\to 0}(1+3x)}=\sqrt{1}=1.$$

$\square$

Exercise 2.7. Is there a real number a such that the following limit exists?

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$$

If so, find the value of a and the value of the limit.

Solution. Yes, the limit exists if $a=15$. If $x\neq -2$ and $x\neq 1$, we have

$$\frac{3x^2+ax+a+3}{x^2+x-2}=\frac{(x+2)(3x+a-6)-a+15}{(x+2)(x-1)}.$$

If $a=15$, this becomes

$$\frac{3x^2+ax+a+3}{x^2+x-2}=\frac{(x+2)(3x+9)}{(x+2)(x-1)}=\frac{3x+9}{x-1}.$$

So, if $a=15$, we have

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}=\lim_{x\to -2}\frac{3x+9}{x-1}=\frac{3}{-3}=-1.$$

$\square$

Exercise 2.8. Are the following statements true or false?

(a) If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \frac{f(x)}{g(x)}$ does not exist.
(b) If x is a real number, then $\sqrt{x^2} = x$
(c) If $\lim_{x\to 5} f(x) = 2$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \frac{f(x)}{g(x)}$ does not exist.
(d) If f is continuous at 5 and $f(5) = 2$, then $\lim_{x\to 2} f(4x^2 - 11) = 2$.
(e) If $f(x) > 1$ for all $x \neq 0$ and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

Solution.

(a) False. If $f(x) = g(x) = x - 5$, then $\lim_{x\to 5} f(x)/g(x) = 1$ while $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$.
(b) False. $\sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1$.
(c) True. f is getting close to 2 while g is getting close to 0, so f/g cannot get close to any finite number.
(d) True. Since f is continuous, $\lim_{x\to 2} f(4x^2 - 11) = f(\lim_{x\to 2}(4x^2 - 11)) = f(5) = 2$.
(e) False. If $f(x) = x^2 + 1$, then $f(x) > 1$ for all $x \neq 0$, but $\lim_{x\to 0} f(x) = 1$.

□

Exercise 2.9. Fix $x \in \mathbf{R}$, and let $f(x) = x^2$. Calculate the following limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

The fraction $(f(x+h)-f(x))/h$ is known as a difference quotient. The limit of this difference quotient will come up again later in the course.

Solution. Note that if $h \neq 0$ we have

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x + h.$$

So, $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2x + h) = 2x$. □

Exercise 2.10. Let $f, g\colon \mathbf{R} \to \mathbf{R}$ and let $a \in \mathbf{R}$. Is it always true that $\lim_{x\to a}(f(x)+g(x)) = (\lim_{x\to a} f(x)) + (\lim_{x\to a} g(x))$?

Solution. No, this is not always true. Consider $f(x) = 1/x$ and $g(x) = -1/x$. Then $\lim_{x\to 0}(f(x) + g(x)) = 0$, but $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist. □

Exercise 2.11. Find all values of $a, b \in \mathbf{R}$ such that the following function is continuous.

$$f(x) = \begin{cases} ax - b & , x \leq -1 \\ 2x^2 + 3ax + b & , -1 < x \leq 1 \\ 4 & , x > 1 \end{cases}.$$

Solution. We show that $a = 3/4$, $b = -1/4$.

By the definition of f, f is a polynomial on the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. That is, f is continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. So, in order for f to be continuous on the entire real line, it suffices to show that $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$. We therefore have the following two equations

$$a(-1) - b = 2 + 3a(-1) + b, \qquad \text{and} \qquad 2 + 3a + b = 4.$$

That is, $-2a+2b = -2$ and $3a+b = 2$, i.e. $a-b = 1$ and $3a+b = 2$. Adding these equations gives $4a = 3$, so that $a = 3/4$. Substituting this equality into $a - b = 1$ gives $b = -1/4$. $\square$

Exercise 2.12. For what values of x is the function $g(x) = (3x^5 + 10)^{1/3}$ continuous?

Solution. The function g is continuous for all $x \in \mathbf{R}$. Let $f(x) = x^{1/3}$ and let $k(x) = 3x^5+10$. Then $g(x) = f(k(x))$. Also, k is a polynomial, so it is continuous with domain all real numbers. And f is continuous with domain all real numbers. Also, the domain of f contains the range of k. Therefore, g has domain all real numbers, and it is continuous since it is the composition of 2 continuous functions. $\square$

Exercise 2.13. Draw the following set and describe it in words: the set of all points (x, y) in the plane such that

$$\lim_{t\to\infty} (|x|^t + |y|^t) < 4.$$

Solution. This set is the square in the plane centered at the origin $(0, 0)$ with side length 2. That is, it is the set

$$\{(x, y) \in \mathbf{R}^2 \colon\ |x| \leq 1 \text{ and } |y| \leq 1\}.$$

To see this, note that $\lim_{t\to\infty} |x|^t = \infty$ if $|x| > 1$. Similarly, $\lim_{t\to\infty} |y|^t = \infty$ if $|y| > 1$. So, in order to be in the set where $\lim_{t\to\infty}(|x|^t + |y|^t) < 4$, we must have $|x| \leq 1$ and $|y| \leq 1$. Moreover, in the case $|x| \leq 1$ and $|y| \leq 1$, we have $|x|^t + |y|^t \leq 2 < 4$ for all $t \geq 1$. $\square$

3. Quiz 3

Exercise 3.1. Find

$$\lim_{y\to\infty} \frac{4y^5 + 5}{(y^2 - 2)(2y^2 - 1)}.$$

Solution. This limit is ∞. Let $y \neq 0$. Then

$$\frac{4y^5 + 5}{(y^2 - 2)(2y^2 - 1)} = \frac{4y^5 + 5}{2y^4 - 5y^2 + 2} = \frac{4y^5 + 5}{2y^4 - 5y^2 + 2}\frac{y^{-4}}{y^{-4}} = \frac{4y + \frac{5}{y^4}}{2 - \frac{5}{y^2} + \frac{2}{y^4}}.$$

So, letting $y \to \infty$, the bottom limit is 2, so

$$\lim_{y\to\infty} \frac{4y^5 + 5}{(y^2 - 2)(2y^2 - 1)} = \frac{1}{2} \lim_{y\to\infty} (4y + 5/y^4) = \infty.$$

$\square$

Exercise 3.2. Find all horizontal and vertical asymptotes of the following two functions.

(a) $y = 1/x$.
(b) $y = \sqrt{x^2 + x + 1} - \sqrt{x^2 + x}$.

(A vertical asymptote occurs at a when any one-sided limit of y at a is ∞ or $-\infty$. A horizontal asymptote occurs if $\lim_{x\to\infty} y(x)$ exists, or if $\lim_{x\to-\infty} y(x)$ exists.)

Solution.

(a) The function $y(x)$ has a vertical asymptote at $x = 0$ and a horizontal asymptote at 0. This follows since $\lim_{x\to0^+} y(x) = \infty$, $\lim_{x\to\infty} y(x) = 0$ and $\lim_{x\to-\infty} y(x) = 0$. Also, y is continuous at all nonzero points, so these are the only asymptotes.

(b) The function $y(x)$ has no vertical asymptotes, though $y(x)$ has a horizontal asymptote at 0. We first discuss the horizontal asymptotes. First, note that the domain of $y(x)$ is $(-\infty, 0] \cup [1, \infty)$. Specifically, by the quadratic formula, $x^2 + x + 1$ has no real roots, and it has value 1 at $x = 0$. Therefore, $x^2 + x + 1 > 0$ for all $x \in \mathbf{R}$. Also, $x^2 - x = x(x - 1)$ which is nonnegative if and only if: $x \leq 0$ or $x \geq 1$.

Now, for $x \in (-\infty, 0] \cup [1, \infty)$, write

$$\begin{aligned}\sqrt{x^2+x+1} - \sqrt{x^2+x} &= (\sqrt{x^2+x+1} - \sqrt{x^2+x})\frac{\sqrt{x^2+x+1} + \sqrt{x^2+x}}{\sqrt{x^2+x+1} + \sqrt{x^2+x}} \\ &= \frac{(x^2+x+1) - (x^2+x)}{\sqrt{x^2+x+1} + \sqrt{x^2+x}} = \frac{1}{\sqrt{x^2+x+1} + \sqrt{x^2+x}}.\end{aligned}$$

When $x \to \infty$, the denominator becomes arbitrarily large. That is,

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2+x+1} + \sqrt{x^2+x}} = 0.$$

So, 0 is a horizontal asymptote of $y(x)$. Similarly, when $x \to -\infty$, the denominator becomes arbitrarily large. That is,

$$\lim_{x\to-\infty} \frac{1}{\sqrt{x^2+x+1} + \sqrt{x^2+x}} = 0.$$

So, 0 is the only horizontal asymptote of $y(x)$.

We now show that there are no vertical asymptotes of $y(x)$. Recall that y is continuous on its domain, it has finite limits at $+\infty$ and at $-\infty$, and it satisfies $\lim_{x\to0^-} y(x) = 1$ and $\lim_{x\to1^+} y(x) = \sqrt{3} - \sqrt{2}$. So, y is a bounded function, i.e. it has no vertical asymptotes.

$\square$

Exercise 3.3. Let $g(x) = x^{2/3}$.

(a) Show that $g'(0)$ does not exist.
(b) If $a \neq 0$, find $g'(a)$.
(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
(d) Illustrate part (c) by graphing $y = x^{2/3}$.

(a) Let $x = 0$ and let $h \neq 0$. Then

$$\frac{g(x+h) - g(x)}{h} = \frac{h^{2/3}}{h} = h^{-1/3}.$$

So, $g'(0) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} h^{-1/3}$, and the latter limit does not exist.

(b) For $a \neq 0$, we show that $g'(a) = (2/3)a^{-1/3}$.

Proof. Let $a \neq 0$. Recall that $c^3 - d^3 = (c-d)(c^2+cd+d^2)$. Using $c = (a+h)^{2/3}$ and $d = a^{2/3}$, we get the formula $(a+h)^2 - a^2 = ((a+h)^{2/3} - a^{2/3})((a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3})$. Observe

$$\begin{aligned}
\frac{(a+h)^{2/3} - a^{2/3}}{h} &= \frac{(a+h)^{2/3} - a^{2/3}}{h} \cdot \frac{(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}}{(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}} \\
&= \frac{(a+h)^2 - a^2}{h[(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}]} \\
&= \frac{a^2 + 2ah + h^2 - a^2}{h[(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}]} \\
&= \frac{2a + h}{(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}}.
\end{aligned}$$

So, letting $h \to 0$ and using our limit laws, we get

$$\begin{aligned}
\frac{2}{3}\frac{1}{a^{1/3}} &= \frac{2a}{a^{4/3} + a^{4/3} + a^{4/3}} = \frac{\lim_{h\to 0}(2a+h)}{\lim_{h\to 0}((a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3})} \\
&= \lim_{h\to 0} \frac{2a+h}{(a+h)^{4/3} + (a+h)^{2/3}a^{2/3} + a^{4/3}} = \lim_{h\to 0} \frac{(a+h)^{2/3} - a^{2/3}}{h} = g'(a).
\end{aligned}$$

□

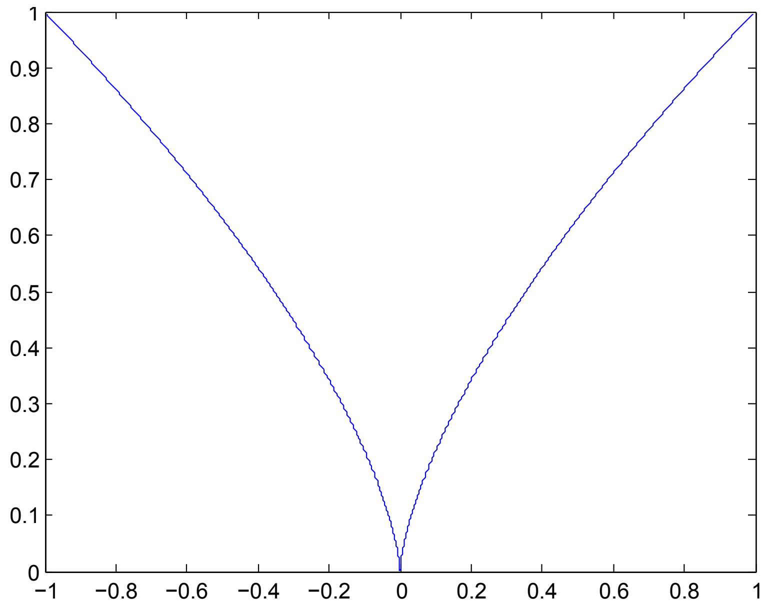
(c) From part (b), $g'(a) = (2/3)a^{-1/3}$. So, $\lim_{a\to 0} |g'(a)| = (2/3)\lim_{a\to 0} |a|^{-1/3} = \infty$.

(d)

Exercise 3.4. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s(t) = 80t - 16t^2$.

(a) What is the maximum height reached by the ball?
(b) What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

(a) The maximum height of the ball is 100 feet. Note that $s'(t) = 80 - 32t$. Let t such that $s'(t) = 0$. Then $80 - 32t = 0$, i.e. $t = 5/2$. For $t \in (0, 5/2)$, $g'(t) > 0$, i.e. the ball is rising in the air. For $t > 5/2$, $g'(t) < 0$, i.e. the ball is falling in the air. So, the maximum height is reached when $t = 5/2$, i.e. when the height of the ball is $s(5/2) = 80(5/2) - 16(5/2)^2 = 200 - 100 = 100$.



(b) The velocity at 96 ft on the way up is 16 ft/s, and the velocity at 96 ft on the way down is -16 ft/s. Let t with $s(t) = 96$. Then $80t - 16t^2 = 96$, so $5t - t^2 = 6$, so $t^2 - 5t + 6 = 0$, i.e. $(t-2)(t-3) = 0$. For $t = 2$, $s(t) = 96$ and $s'(t) = 80 - 64 = 16 > 0$, so the ball is rising at $t = 2$ with velocity 16 ft/s. For $t = 3$, $s(t) = 96$ and $s'(t) = 80 - 96 = -16$, so the ball is falling at $t = 2$ with velocity -16 ft/s.

Exercise 3.5. Suppose the curve $y(x) = x^4 + ax^3 + bx^2 + cx + d$ has a tangent line when $x = 0$ with equation $y = 2x + 1$ and a tangent line when $x = 1$ with equation $y = 2 - 3x$. Find the values of $a, b, c,$ and d.

Solution. We have $(a, b, c, d) = (1, -6, 2, 1)$.

Note that $y'(x) = 4x^3 + 3ax^2 + 2bx + c$. At $x = 0$, $y'(x) = c$, and we have a tangent line with slope 2, so $y'(0) = c = 2$. Also, since the tangent line is $y = 2x + 1$ at $x = 0$, we must have $y(0) = 1$, so that $y(0) = d = 1$. So, we have determined the values of c and d. At $x = 1$, $y'(x) = 4 + 3a + 2b + c = 4 + 3a + 2b + 2 = 6 + 3a + 2b$. Also, at $x = 1$ we have a tangent line with slope -3, so $y'(1) = 6 + 3a + 2b = -3$, i.e. $3a + 2b = -9$. Also, since the tangent line is $y = 2 - 3x$ at $x = 1$, we must have $y(1) = -1$, so that $y(1) = 1 + a + b + c + d = -1$, i.e. $1 + a + b + 2 + 1 = -1$, i.e. $a + b = -5$, i.e. $-2a - 2b = 10$. Adding the equations $3a + 2b = -9$ and $-2a - 2b = 10$, we get $a = 1$. Finally, since $a + b = -5$, we conclude that $b = -6$. $\square$

4. Quiz 4

Exercise 4.1. One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be); k is the cost of placing an order (the cost is the same, no matter how often you make an order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2, and interpret your results in terms of the constants.

Solution. We have $dA/dq = (h/2) - kmq^{-2}$ and $d^2A/dq^2 = 2kmq^{-3}$. Note that when q is large, we order a lot of inventory, so our cost is rising since the weekly cost h begins to affect our cost a lot. □

Exercise 4.2. Find the equation of the tangent to the curve at the given point.

(a) $y = 4(1+x^3)(x+x^{10})$, $(x,y) = (1,16)$, (b) $y = \dfrac{x^2-1}{x^2+1}$, $(x,y) = (0,-1)$.

Solution. (a) From the product rule, $y'(x) = 4(1+x^3)(1+10x^9) + 4(3x^2)(x+x^{10})$, so that $y'(1) = 4(2)(11) + 4(3)(2) = 112$. So, the tangent line is $y = 112x - 96$.

(b) We have $y'(x) = \frac{(x^2+1)(d/dx)(x^2-1)-(x^2-1)(d/dx)(x^2+1)}{(x^2+1)^2} = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$. So, $y'(0) = 0$, and the tangent line is $y = -1$. □

Exercise 4.3. Find the equation of the tangent line to the curve at the given point.

(a) $y = \sqrt{1+4x^2}$, $(x,y) = (0,1)$, (b) $x^2 + 2y^2 = 1$, $(x,y) = (1/\sqrt{2}, -1/2)$.

Solution. (a) We have $y'(x) = (1/2)(1+4x^2)^{-1/2}(8x)$ by the chain rule, so that $y'(0) = (1/2)(8) = 4$. So, the tangent line is $y = 4x + 1$.

(b) Solving for y we have $y = \pm 2^{-1/2}\sqrt{1-x^2}$. Since $-1/2 < 0$, we have to choose the negative sign, so that $y = -2^{-1/2}\sqrt{1-x^2}$. From the Chain Rule, $y'(x) = -2^{-1/2}(1/2)(1-x^2)^{-1/2}(-2x)$, so that $y'(1/\sqrt{2}) = -2^{-1/2}(1/2)2^{-1/2}(-\sqrt{2}) = \sqrt{2}/4$. And the tangent line is $y = (\sqrt{2}/4)x - 3/4$. □

Exercise 4.4. Let $f(x) = x^{2/3}$. Draw f and f' in the same plot.

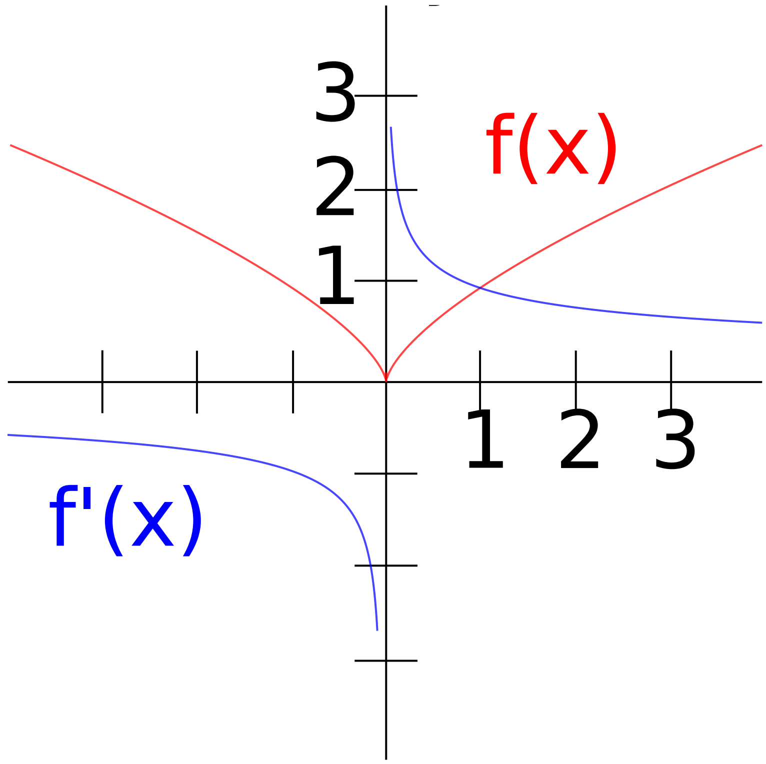
Solution.

□

Exercise 4.5. Let $P(t)$ represent the price of a share of stock of a corporation at time t. What does each of the following statements tell us about the signs of the first and second derivatives of $P(t)$?

- "The price of the stock is rising faster and faster."
- "The price of the stock is close to bottoming out"

Solution. The first statement says $P'(t) > 0$ and $P''(t) > 0$ since the price is increasing at an accelerating rate.



The second statement says $P'(t) < 0$ and $P''(t) > 0$ since the price is decreasing, but the derivative of the price is increasing. □

Exercise 4.6. In economics, total utility refers to the total satisfaction from consuming some commodity. According to the economist Samuelson:

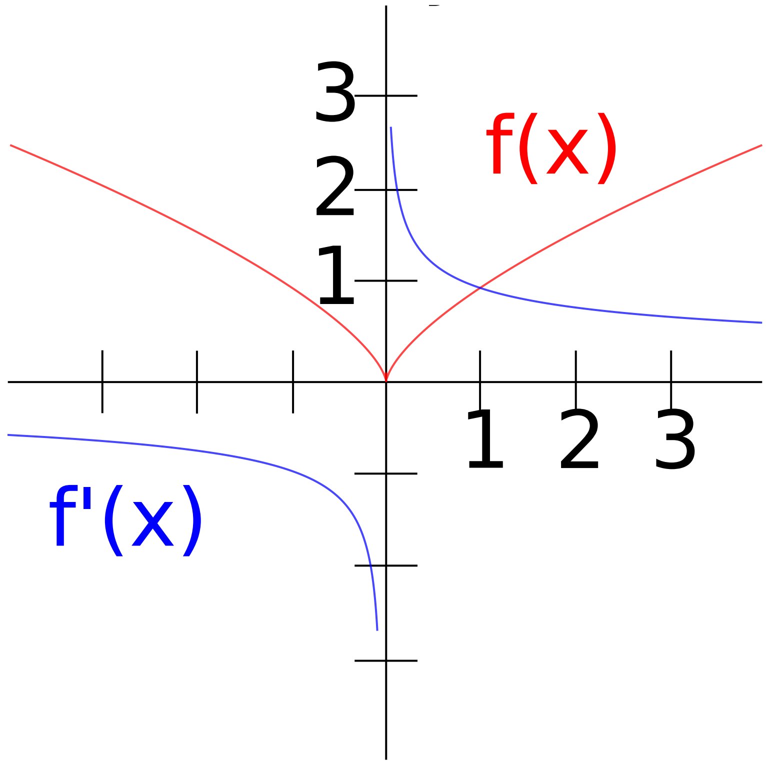
"As you consume more of the same good, the total (psychological) utility increases. However, . . . with successive new units of the good, your total utility will grow at a slower and slower rate because of a fundamental tendency for your psychological ability to appreciate more of the good to become less keen."

- Sketch the total utility as a function of the number of units consumed.
- In terms of derivatives, what is Samuelson saying?

Solution.

Samuelson is saying that the first derivative of utility is positive, but its second derivative eventually becomes negative, similar to the function we drew above for $x^{2/3}$ when $x > 0$.

□



5. Quiz 5

Exercise 5.1. In chemistry, a "second order" reaction satisfies $dy/dt = k(y(t))^2$ where $y(t)$ is the concentration of some chemical at time t. If $y_0 = y(0) \neq 0$, verify that we must have

$$y(t) = \frac{1}{y_0^{-1} - kt}.$$

Solution. Writing $y(t) = (y_0^{-1} - kt)^{-1}$ and using the Chain Rule,

$$y'(t) = -(y_0^{-1} - kt)^{-2}(-k) = k(y(t))^2.$$

□

Exercise 5.2 (Newton's Law of Cooling). Suppose $y(t)$ is the temperature of an object at time t. If an object is of a different temperature than its surroundings, then the rate of change of the object's temperature is proportional to the difference of the temperature of the object and the temperature of the surroundings. That is, if Y denotes the temperature of the surroundings, and if $y(0) = y_0 \neq Y$, then there exists a constant $k > 0$ such that

$$y'(t) = -k(y(t) - Y).$$

Note that if $y(0) < Y$, then $y'(0) > 0$, so that the temperature of y is increasing to the environment's temperature. And if $y(0) > Y$, then $y'(0) < 0$, so that y is decreasing to the environment's temperature.

Let $f(t) = y(t) - Y$. Verify that $f'(t) = -kf(t)$. Conclude that $f(t) = y(t) - Y = (y_0 - Y)e^{-kt}$. That is, we have Newton's Law of cooling:

$$y(t) = Y + (y_0 - Y)e^{-kt}.$$

Solution. Since Y is a constant, $f'(t) = y'(t) = -k(y(t) - Y) = -kf(t)$. The second equality is an assumption of $y'(t)$, and the last equality used the definition of $f(t)$. Since $f'(t) = -kf(t)$, we conclude that $f(t) = ce^{-kt}$ for some constant c. Since $f(0) = y_0 - Y$ by definition of f and $f(0) = ce^0 = c$, we have $c = y_0 - Y$. So, $f(t) = (y_0 - Y)e^{-kt}$. Recalling the definition of $f(t) = y(t) - Y$, we finally, have

$$y(t) = Y + (y_0 - Y)e^{-kt}.$$

□

Exercise 5.3. The exponential growth model for bacteria is a bit unrealistic, since after a while, the bacteria are limited by their environment and food supply. We therefore consider the **logistic growth** model. Suppose $y(t)$ is the amount of bacteria in a petri dish at time t and $k > 0$ is a constant. Let C be the maximum possible population of the bacteria. We model the growth of the bacteria by the formula

$$y'(t) = ky(t)(C - y(t)), \qquad y(0) = y_0$$

So, when y is small, $y'(t)$ is proportional to y. However, when y becomes close to C, y' becomes very small. That is, the rate of growth of bacteria is constrained by the environment.

- Verify that the following function satisfies the above differential equation.
$$y(t) = \frac{C}{1 + (Cy_0^{-1} - 1)e^{-ktC}}.$$
- Plot the function $y(t)$. (What are the limits of y as t goes to $+\infty$ and $-\infty$?)
- Find out where $y'(t)$ is the largest. (Hint: find the maximum of the function of y: $ky(C-y)$.) (This point t is called the point of **diminishing returns** in economics.)

The latter observation explains the "J-curve" scare for human population growth in the 1980s. At this point in time, many people were afraid that the human population would grow too large for the earth to support us. However, it seems that we were simply observing the maximum possible growth rate of the human population at this time (if we believe that logistic growth models the human population reasonably well).

Solution. Writing $y(t) = C[1 + (Cy_0^{-1} - 1)e^{-ktC}]^{-1}$ and using the Chain Rule,

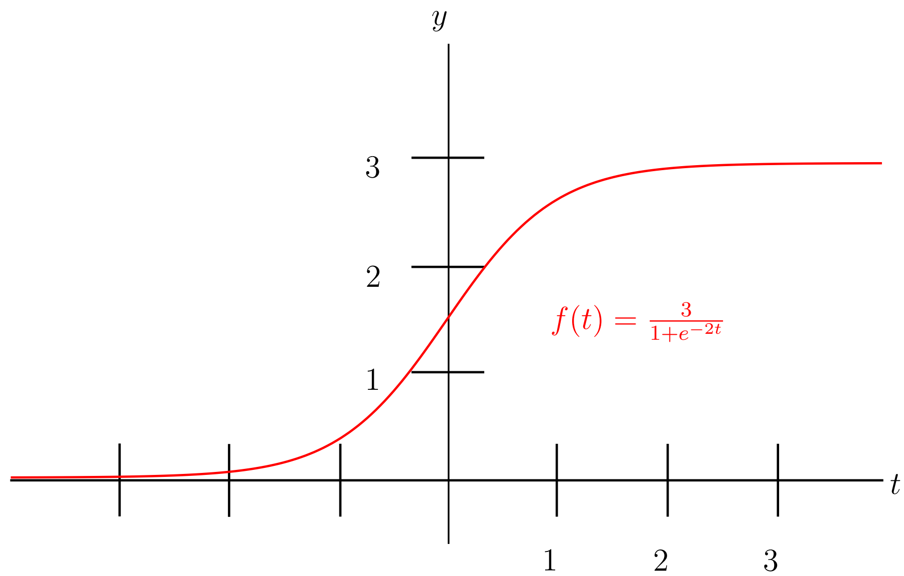
$$y'(t) = -C[1 + (Cy_0^{-1} - 1)e^{-ktC}]^{-2}(Cy_0^{-1} - 1)(-kC)e^{-ktC} = k(y(t))^2(Cy_0^{-1} - 1)e^{-ktC}.$$

Also

$$C - y(t) = \frac{C[1 + (Cy_0^{-1} - 1)e^{-ktC}] - C}{1 + (Cy_0^{-1} - 1)e^{-ktC}} = \frac{C[(Cy_0^{-1} - 1)e^{-ktC}]}{1 + (Cy_0^{-1} - 1)e^{-ktC}} = y(t)(Cy_0^{-1} - 1)e^{-ktC}.$$

So, combining these two lines,

$$y'(t) = ky(y)(C - y(t)).$$



Also, by definition of y,

$$y(0) = \frac{C}{1 + (Cy_0^{-1} - 1)} = \frac{C}{Cy_0^{-1}} = y_0.$$

As a function of y, the quantity $g(y) = ky(C - y) = k(-y^2 + yC)$ is a parabola, and $g'(y) = k(-2y + C)$, so that $g'(y) = 0$ only when $y = C/2$. Since $k > 0$, $g(y) = ky(C - y)$ is a concave down parabola, so it has its absolute maximum at the only critical point $y = C/2$. That is, the point of greatest increase of y occurs when the population is half of its maximum size. □

Exercise 5.4. Let $f(x) = (1 + x)^{15}$. Near $x = 0$, show that the linear approximation of f is given by $1 + 15x$, meaning $(1 + x)^{15} \approx (1 + 15x)$ when x is near zero.

Solution. Since $f'(x) = 15(1 + x)^{14}$, we have $f'(0) = 15$. So, near $x = 0$,

$$f(x) \approx f(0) + f'(0)x = 1 + 15x.$$

□

Exercise 5.5. Let $a, b > 0$. Find the maximum value of $f(x) = x^a(1 - x)^b$ on the interval $[0, 1]$.

Solution. The maximum value is

$$f(a/(a + b)) = (a/(a + b))^a(1 - a/(a + b))^b = a^a b^b/(a + b)^{a+b}.$$

Observe

$$\begin{aligned} f'(x) &= -x^a b(1 - x)^{b-1} + ax^{a-1}(1 - x)^b \\ &= x^{a-1}(1 - x)^{b-1}(-bx + a(1 - x)) = x^{a-1}(1 - x)^{b-1}(a - x(a + b)). \end{aligned}$$

So, the critical points of f are $x = 0, 1, a/(a + b)$. Note that f is continuous on the interval $[0, 1]$, since it is a product of continuous functions. So, by the Closed Interval Method, the maximum value of f must occur at its critical points. Since $f(0) = f(1) = 0$, and since $f(x) > 0$ for $0 < x < 1$, we conclude that the maximum value of f occurs at the point $x = a/(a+b)$. □

Exercise 5.6. Let $a, b > 0$. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$. You may assume that the rectangle is aligned with the axes, and that its vertices touch the ellipse.

Solution. The largest area rectangle has area $2ba$.

Suppose the upper right point of the rectangle has coordinates (x, y) with $x, y > 0$, $-a \leq x \leq a$, $-b \leq y \leq b$. Then $y^2 = b^2(1 - x^2/a^2)$. Since $y > 0$, we have $y = b\sqrt{1 - x^2/a^2}$. Then the rectangle has width $2x$ and height $2b\sqrt{1 - x^2/a^2}$, so its volume $V(x)$ is given by

$$V(x) = 4bx\sqrt{1 - x^2/a^2}.$$

We therefore try to maximize $V(x)$ on the interval $[0, a]$. Note that $V(x)$ is a continuous function of x on the interval $[0, a]$, since it is the composition of continuous functions. So, by the Closed Interval Method, it suffices to check the value of V on the endpoints of the interval $[0, a]$ along with the critical points of V.

For $x \in (0, a)$, note that

$$\begin{aligned} V'(x) &= -4ba^{-2}x^2(1 - x^2/a^2)^{-1/2} + 4b(1 - x^2/a^2)^{1/2} \\ &= (1 - x^2/a^2)^{-1/2}(-4bx^2/a^2 + 4b(1 - x^2/a^2)). \end{aligned}$$

So, $V'(x) = 0$ only when $-x^2(2/a^2) + 1 = 0$, i.e. when $x^2 = a^2/2$. Since $x > 0$, we conclude that $x = a/\sqrt{2}$. Finally, since $V(0) = V(a) = 0$, and since $V(x) > 0$ for $0 < x < a$, we conclude that $V(x)$ is maximized on the interval $[0, a]$ when $x = a/\sqrt{2}$. □

Exercise 5.7. Find the minimum and maximum values of $f(x) = 2\sqrt{x^2 + 1} - x$ on the interval $[0, 2]$

Solution. We have $f'(x) = 2x\sqrt{x^2 + 1} - 1$. If $f'(x) = 0$, we have $2x(x^2 + 1)^{-1/2} = 1$, so that $2x = (x^2 + 1)^{1/2}$, and $4x^2 = x^2 + 1$, i.e. $3x^2 = 1$, i.e. $x = \pm 1/\sqrt{3}$. Since we are only considering $x \in [0, 2]$, we see that $x = 1/\sqrt{3}$ is the only critical point of f. So, the minimum and maximum values must occur among the points: $0, 1/\sqrt{3}, 2$. We check that $f(0) = 2$, $f(2) = 2\sqrt{5} - 2 = 2(\sqrt{5} - 1)$ and $f(1/\sqrt{3}) = 2\sqrt{1/3 + 1} - 1/\sqrt{3} = 3/\sqrt{3} = \sqrt{3}$. Since $\sqrt{3} < 2 < 2(\sqrt{5} - 1)$, the minimum value of f is $\sqrt{3}$ and the maximum value of f is $\sqrt{5} - 1$. □

Exercise 5.8. It is the zombie apocalypse. You are in the forest, five miles from a straight road. If you traveled in a straight line towards the road from your current position (which is not a good idea), you would then have to walk 10 miles along the road to get to the safe house. You can travel at two miles per hour in the forest, and you can travel at four miles per hour on the road. What is the shortest amount of time that it will take to get to the safe house at the end of the road?

Solution. Let $0 \leq x \leq 10$ so that $10 - x$ is the distance you travel on the road. Using right triangles, the distance you travel in forest is $\sqrt{5^2 + x^2} = \sqrt{25 + x^2}$. The total travel time is then $f(x) = \sqrt{25 + x^2}/2 + (10 - x)/4$. Taking the derivative in x, we get $x(25 + x^2)^{-1/2}/2 - 1/4$. Solve, the equation $f'(x) = 0$, we get $1/(2x) = (25 + x^2)^{-1/2}$, so that $2x = \sqrt{25 + x^2}$, i.e. $4x^2 = 25 + x^2$, i.e. $3x^2 = 25$, so that $x = \sqrt{25/3}$. (Note that $x = -\sqrt{25/3}$ corresponds to traveling away from the safe house.) Plugging this back into f, we get a total travel time of $\sqrt{25 + 25/3}/2 + (10 - \sqrt{25/3})/4 \approx 4.67$ hours when $x = \sqrt{5}$. Comparing this to the endpoints of the interval, we have $f(0) = 5/2 + 10/4 = 5$ hours (which corresponds to walking directly towards the road, and then using the road), and $f(10) = \sqrt{125}/2 = 5\sqrt{5}/2 \approx 5.6$ hours (which corresponds to walking directly towards the safe house). So, the fastest travel time is $\sqrt{30}/2 + (10 - \sqrt{5})/4$ hours. □

6. Quiz 6

Exercise 6.1. (**A speeding ticket?**) Suppose I am driving in a car, and there are police cameras that are stationed at certain mile markers. The first camera spots my license plate at 10 AM. Five miles down the road, the second camera spots my license plate at 10 : 04 AM. If my speed exceeded 74 miles per hour at any particular point in time, I will automatically be issued a ticket in the mail. Will I be issued a ticket?

Solution. Let $s(t)$ be the position of the car at time t, so that $s'(t)$ is the speed of the car. From the Mean Value Theorem, there exists some time T between 10 AM and 10 : 04 AM such that $s'(T) = (s(10:04) - s(10))/(10:04 - 10) = 5/4$ miles per minute. So, $s'(T) = 75$ miles per hour. So, yes, I will get a ticket, since I exceeded 74 mph at some point in time. □

Exercise 6.2. Let $f(x) = x^5 - 5x^3/3 + 1$. Find the critical points of f, and find the intervals where f is increasing and decreasing. Apply the first derivative test to each critical point.

Solution. We have $f'(x) = 5x^4 - 5x^2 = 5x^2(x^2 - 1)$. So, $f'(x) = 0$ when $x = -1, 0, 1$. We have $f'(x) > 0$ when $x < -1$; $f'(x) < 0$ when $-1 < x < 1$; and $f'(x) > 0$ when $x > 1$. So, f is increasing when $x < -1$, f is decreasing when $-1 < x < 1$, and f is increasing when $x > 1$. From the first derivative test, we see that $x = -1$ is a local maximum, $x = 1$ is a local minimum, and $x = 0$ is not a local extremum. □

Exercise 6.3. Let $f(x) = x^4/4 - x^2/2$. Find the critical points of f, and find the intervals where f is increasing and decreasing. Apply the first derivative test to each critical point.

Solution. We have $f'(x) = x^3 - x = x(x + 1)(x - 1)$. So, $f'(x) = 0$ when $x = -1, 0, 1$. We have $f'(x) < 0$ when $x < -1$; $f'(x) > 0$ when $-1 < x < 0$; and $f'(x) < 0$ when $0 < x < 1$ and $f'(x) > 0$ when $x > 1$. So, f is decreasing when $x < -1$, f is increasing when $-1 < x < 0$, f is decreasing when $0 < x < 1$, and f is increasing when $x > 1$. From the first derivative test, we see that $x = -1$ is a local minimum, $x = 0$ is a local maximum, and $x = 1$ is a local minimum. □

Exercise 6.4. Find two numbers whose difference is 100 and whose product is a minimum.

Solution. Let x, y be the two numbers. We have $x - y = 100$, so that $x = 100 + y$, and we need to minimize xy. Define $f(y) = xy = (100 + y)y = 100y + y^2$. Then $f'(y) = 100 + 2y$. so, $f'(y) = 0$ when $y = -50$. When $y = -50$, we have $x = 50$. Note that $f'(y) < 0$ when $y < -50$ and $f'(y) > 0$ when $y > -50$. So, $y = -50$ is the global minimum of f. The two numbers are therefore 50 and -50. □

Exercise 6.5. Suppose 1200 cm^2 of material is available to make a box with a square base and an open top. Find the largest possible volume of the box.

Solution. Suppose the square has side length ℓ and the box has height h. The area of the box is then $\ell^2 + 4\ell h$. We have $1200 = \ell^2 + 4\ell h$. The volume of the box is $\ell^2 h$. We have $h = (1200 - \ell^2)/(4\ell)$. So, the volume of the box is $f(\ell) = \ell^2 h = \ell(1200 - \ell^2)/4 = 300\ell - \ell^3/4$. Then $f'(\ell) = 300 - 3\ell^2/4$. So, $f'(\ell) > 0$ then $0 \leq \ell < 20$, and $f'(\ell) < 0$ when $\ell > 20$. So, the global maximum of $f(\ell)$ occurs when $\ell = 20$. At this point, we have $f(\ell) = 20(1200 - 400)/4 = (20)(800)/4 = 4000$. □

Exercise 6.6. Find the point on the line $y = 2x + 3$ that is closest to the origin.

Solution. We need to minimize the quantity $\sqrt{x^2 + y^2} = \sqrt{x^2 + (2x + 3)^2}$. Equivalently, we need to minimize the quantity $f(x) = x^2 + (2x + 3)^2 = 5x^2 + 12x + 9$ over all $x \in \mathbf{R}$. We have $f'(x) = 10x + 12$, so $f'(x) < 0$ when $x < -6/5$ and $f'(x) > 0$ when $x > -6/5$. So, $x = -6/5$ is the global minimum of f. So, the closest point on this line to the origin is the point $(-6/5, 3/5)$. □

Exercise 6.7. For a fish swimming at a speed v relative to the water, the energy expenditure per unit of time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current of speed u ($u < v$), then the time required to swim a distance L is $L/(v - u)$, and the total energy E required to swim the distance L is given by

$$E(v) = av^3 \frac{L}{v - u}$$

Here a is an arbitrary constant.

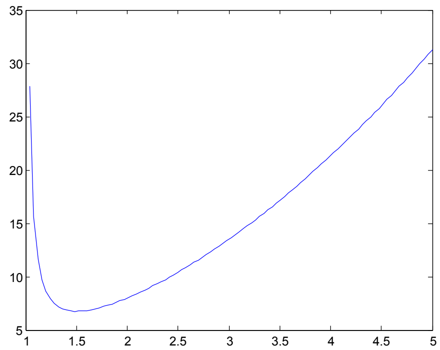
(a) Determine the value of v that minimizes E.
(b) Sketch the graph of E.

Note: This result has been verified experimentally. Migrating fish swim against a current at a speed 50% greater than the speed of the current.

Solution. We have $E'(v) = aL\frac{(v-u)3v^2 - v^3}{(v-u)^2} = aL\frac{2v^3 - 3uv^2}{(v-u)^2}$. So, $E'(v) = 0$ when $2v^3 - 3uv^2 = 0$. So, either $v = 0$ or $2v = 3u$, i.e. $v = 3u/2$. (Also, $E'(v)$ is undefined when $v = u$, but we are only considering $v > u$.) Note that $E'(v) < 0$ when $u < v < 3u/2$ and $E'(v) > 0$ when $v > 3u/2$. So, $v = 3u/2$ is the global minimum of $E(v)$ for $v > u$.

Here is a plot of E when $a = L = u = 1$. □

Exercise 6.8. A cabinetmaker uses mahogany to produce 5 furnishings each day. Each delivery of one container of wood costs \$5000, and storage of that material is \$10 per day per



unit stored, where a unit is the amount of material needed by her to produce 1 furnishing. How much material should be ordered each time and how often should the material be delivered to minimize her average daily cost in the production cycle between deliveries? (You can consider one container of wood to have an unlimited capacity, and the storage cost of one day is equal to the number of units of wood in the shop at the beginning of the day.)

Solution. Suppose the number of days between deliveries is x. Once a delivery has been made and right before the next delivery, the cabinetmaker spends \$5000 for the first wood delivery plus 10 times $(5+10+15+\cdots+5x) = 5(1+2+3+\cdots+x) = 5x(x+1)/2$. (It is most efficient to run out of wood and have the wood delivered right after the wood runs out. So, on the last day before the next delivery, there are 5 units of wood in the shop, which have a storage cost of $10\cdot 5$; the day before, the storage cost was $10\cdot 10$; the day before that, the storage cost was $10\cdot 15$, and so on. On the day of the delivery, the mahogany was replenished, so there were $5x$ units in the shop, and there was a storage cost of $10\cdot 5x$).

So, the cost for the time before the next delivery is $5000+5(5x)(x+1)$. The average daily cost is then $f(x) = 5000/x + 25(x+1)$. We have $f'(x) = -5000x^{-2}+25$. If $f'(x) = 0$, we have $25 = 5000x^{-2}$. That is, $x^2 = 5000/25 = 200$ or $x = 10\sqrt{2}$ (since $x > 0$). So, $x \approx 14.14$. Note that $f'(14) < 0$, $f'(15) > 0$, and f' is increasing. So, the minimum value of f occurs somewhere between 14 and 15. Since x is an integer, the maximum value of f when x is an integer occurs when $x = 14$ or $x = 15$. We therefore check that $f(14) = 5000/14 + 25(15) \approx 732$ and $f(15) = 5000/15 + 25(16) \approx 733$. So, the smallest average cost occurs when $x = 14$. In the case $x = 14$, we are ordering $5\cdot 14 = 70$ units every 14 days. □

Exercise 6.9. (**How we cough**) When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity of air, and whether the trachea really contracts that much when we cough.

Let r_0 be the rest radius of the trachea in centimeters, and let c be a positive constant whose value depends in part on the length of the trachea. Under reasonable assumptions about how the air near the wall is slowed by friction, the average air flow velocity v can be modeled by the equation

$$v = c(r_0 - r)r^2 \ \text{cm/sec}, \qquad \frac{r_0}{2} \le r \le r_0.$$

Show that v is greatest when $r = (2/3)r_0$. That is, the velocity is greatest when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

Solution. We have $v'(r) = -3cr^2 + 2crr_0 = cr(-3r + 2r_0)$. So, $v'(r) = 0$ when $-3r + 2r_0 = 0$, i.e. when $r = 2r_0/3$. Also, $v'(r) > 0$ when $0 \leq r < 2r_0/3$ and $v'(r) < 0$ when $r > 2r_0/3$. So, $2r_0/3$ is the absolute maximum of v. □

Exercise 6.10. The average car achieves its best fuel efficiency at a speed of around 50 or 55 miles per hour. Driving faster than this speed can drastically reduce fuel efficiency, as we now show.

The fuel efficiency $f(v)$ of a car (in miles per gallon) traveling at a velocity v (in miles per hour) can be roughly modelled as

$$f(v) = 200,000 \frac{v}{v^3 + 2 \cdot 50^3}$$

When v is small, resistance from the tires, the mechanical aspects of the engine, etc. contribute to the inefficiency. When v is large, resistance from air also becomes a factor.

Find the maximum fuel efficiency of the car. Compare this efficiency to the values $v = 60$, $v = 70$ and $v = 80$. (Due to air friction and other frictional forces, driving faster is often less efficient.)

Solution. We have $f'(v) = [(v^3 + 2 \cdot 50^3) - v(3v^2)]/(v^3 + 2 \cdot 50^3)^2$. So, $f'(v) = 0$ when $-2v^3 + 2 \cdot 50^3 = 0$, i.e. when $v = 50$. Note that $f(50) = 200000(50)/(50^3 + 2 \cdot 50^3) \approx 26.7$, while $f(60) \approx 25.8$, $f(70) \approx 23.6$, $f(80) \approx 21.0$. □

7. Quiz 7

Exercise 7.1. Let $f\colon [0, 8] \to \mathbf{R}$ be a function such that $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, $f(3) = 2$, $f(4) = 0$, $f(5) = 6$, $f(6) = 1$, $f(7) = 2$ and $f(8) = 0$. Using four equally sized rectangles, find the Riemann sums of f evaluated at the right endpoints, evaluated at the left endpoints, and evaluated at the midpoints.

Solution. The Riemann sum evaluated at the left endpoints is

$$2(f(0) + f(2) + f(4) + f(6)) = 2(1 + 4 + 0 + 1) = 12.$$

The Riemann sum evaluated at the right endpoints is

$$2(f(2) + f(4) + f(6) + f(8)) = 2(4 + 0 + 1 + 0) = 10.$$

The Riemann sum evaluated at the midpoints is

$$2(f(1) + f(3) + f(5) + f(7)) = 2(2 + 2 + 6 + 2) = 24.$$

□

Exercise 7.2. Evaluate
$$\lim_{n\to\infty}\frac{1^5+2^5+\cdots+n^5}{n^6},$$
by showing that the limit is $\int_0^1 x^5dx$.

Solution. Let $f(x)=x^5$ and let $x_i=i/n$ for all $0\leq i\leq n$. Then
$$\int_0^1 f(x)dx=\lim_{n\to\infty}\sum_{i=1}^n(x_i-x_{i-1})f(x_i)=\lim_{n\to\infty}\sum_{i=1}^n(1/n)(i/n)^5=\lim_{n\to\infty}\frac{1^5+2^5+\cdots+n^5}{n^6}.$$
Since $\int_0^1 x^5dx=[x^6/6]_{x=0}^{x=1}=1/6$, we have $\lim_{n\to\infty}\frac{1^5+2^5+\cdots+n^5}{n^6}=1/6$. □

Exercise 7.3. Let $f\colon\mathbf{R}\to\mathbf{R}$ be continuous with two continuous derivatives. Find all such f such that $f''(x)=20x^3-12x^2+6x$.

Solution. We must have $f(x)=x^5-x^4+x^4/2+Cx+D$ where C,D are constants. Note that this function f satisfies $f''(x)=20x^3-12x^2+6x$. □

Exercise 7.4. Two balls are thrown upward from the edge of a cliff of height 432 feet. The first ball is thrown upward with a speed of 48 ft/s, and the other ball is thrown upward a second later with a speed of 24 ft/s. Do the balls ever pass each other?

Solution. Yes, the balls pass each other five seconds after the first ball is thrown.

Let t denote time in seconds, with $t=0$ denoting the time that the first ball is thrown. Let $a_1(t),v_1(t),s_1(t)$ denote the acceleration, velocity, and position of the first ball, and let $a_2(t),v_2(t),s_2(t)$ denote the acceleration, velocity, and position of the second ball. All units use feet and seconds. It is given that $a_1=a_2=-32$, $v_1(0)=48$, $v_2(1)=24$, $s_1(0)=432$, and $s_2(1)=432$. Since $v_1'(t)=a_1(t)$, and since all trajectories are assumed to be continuous and differentiable, we know that $v_1(t)=-32t+c$ for some $c\in\mathbf{R}$. Then $v_1(0)=48=c$, so $v_1(t)=-32t+48$. Since $s_1'(t)=v_1(t)$, we conclude that $s_1(t)=-16t^2+48t+d$ for some $d\in\mathbf{R}$. Then $s_1(0)=432=d$, so
$$s_1(t)=-16t^2+48t+432.$$

We now again use antiderivatives for the trajectory of the second ball. Since $v_2'(t)=a_2(t)$, we have $v_2(t)=-32t+e$ for some $e\in\mathbf{R}$. Then $v_2(1)=24=-32+e$, so $e=56$, so $v_2(t)=-32t+56$. Since $s_2'(t)=v_2(t)$, we have $s_2(t)=-16t^2+56t+g$ for some $g\in\mathbf{R}$. And $s_2(1)=432=-16+56+g$, so $g=392$, so
$$s_2(t)=-16t^2+56t+392.$$

We now check whether or not the balls pass each other. Let $f(t)=s_1(t)-s_2(t)$. To complete the exercise it suffices to find $t>0$ where $f(t)$ changes sign. Suppose t satisfies $f(t)=0$. Then $48t+432-56t-392=0$, so $-8t=-40$, so $t=5$. So, $f(5)=0$. Also, note that $s_1(5)=s_2(5)=272>0$, so at $t=5$, both balls are still above the ground, and they have the same height above the ground. We now check the derivative of f at 5. Note that $f'(t)=48-56=-8<0$. In conclusion, for $0<t<5$, $s_1(t)-s_2(t)=f(t)>0$, and for $t>5$, $s_1(t)-s_2(t)=f(t)<0$. So, the balls do pass each other at the time $t=5$. □

Exercise 7.5. Let $a < b$ and let m, M be constants. For a continuous function f, we know from Property (9) for integrals that if $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Use this property to estimate $\int_0^2 (x^3 - 3x + 3)dx$.

Solution. Let $f(x) = x^3 - 3x + 3$. Then $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$. So, f is decreasing on $[0, 1]$ and increasing on $[1, 2]$. So, the maximum and minimum values of f on $[0, 2]$ occur at $x = 0, 1, 2$. We have $f(0) = 3$, $f(1) = 1$ and $f(2) = 5$. So, $1 \leq f(x) \leq 5$ for all $x \in [0, 1]$. Therefore,

$$2 = 1(2-0) \leq \int_0^2 x^3 - 3x + 3dx \leq 5(2-0) = 10.$$

□

Exercise 7.6. Let $f, g \colon \mathbf{R} \to \mathbf{R}$ be integrable functions. Suppose $\int_0^9 f(x)dx = 5$ and $\int_0^9 g(x)dx = 7$. Find $\int_0^9 (3f(x) + 2g(x))dx$.

Solution. $\int_0^9 (3f(x) + 2g(x))dx = 3(\int_0^9 f(x)dx) + 2(\int_0^9 g(x)dx) = 3(5) + 2(7) = 29$. □

Exercise 7.7. Using the Fundamental Theorem of Calculus, evaluate $\int_{-2}^3 (x^2 - 3)dx$.

Solution.

$$\begin{aligned}\int_{-2}^3 (x^2 - 3)dx &= \int_{-2}^3 \frac{d}{dx}(x^3/3 - 3x)dx = [x^3/3 - 3x]_{x=-2}^{x=3} \\ &= 9 - 9 - (-2)^3/3 + 3(-2) = -8/3 - 6 = -26/3.\end{aligned}$$

□

Exercise 7.8. Using the Fundamental Theorem of Calculus, evaluate $\int_3^5 (x^3 + x^{-2} + e^x)dx$.

Solution.

$$\begin{aligned}\int_3^5 (x^3 + x^{-2} + e^x)dx &= \int_3^5 \frac{d}{dx}(x^3/3 - x^{-1} + e^x)dx = [x^3/3 - x^{-1} + e^x]_{x=3}^{x=5} \\ &= 5^3/3 - 1/5 + e^5 - 9 + 1/3 - e^3.\end{aligned}$$

□

8. Quiz 8

Exercise 8.1. State whether or not the statement is True or False. Justify your answer. Let $a < b$.

(1) If $f, g\colon [a,b] \to \mathbf{R}$ are continuous, then
$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$
(2) If $f, g\colon [a,b] \to \mathbf{R}$ are continuous, then
$$\int_a^b (f(x)g(x))dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right).$$
(3) If $f, g\colon [a,b] \to \mathbf{R}$ are continuous, and if $f(x) \geq g(x)$ for all $x \in [a,b]$, then
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$
(6) If $f\colon (a,b) \to \mathbf{R}$ is continuous, then $\int_a^b f(x)dx$ exists.

Solution.

(1) True. We stated this as a property of integrals.
(2) False. Consider f which is 1 on $[0,1]$ and zero otherwise, and consider g which is 1 on $[-1,0]$ and zero otherwise. Then $\int_{-1}^1 f(x)g(x) = 0$, but $(\int_{-1}^1 f(x)dx)(\int_{-1}^1 g(x)dx) = 1$
(3) True. We stated this as a property of integrals.
(4) False. $f(x) = 1/x$ is continuous on $(0,1)$ but $\int_0^1 f(x)dx$ does not exist.

□

Exercise 8.2. Suppose we are manufacturing n shoes. The average cost $AC(n)$ of producing n shoes is the total cost of producing all n shoes, divided by n. Assume that
$$AC(n) = \frac{100}{n} + 50 + 5n^2.$$

- What is the total cost $C(n)$ of producing n shoes?
- What is the fixed cost? (The fixed cost is the constant term in $C(n)$, i.e. the part of $C(n)$ that does not depend on n.)
- What is the marginal cost?

Solution. By definition of $AC(n)$, we have $\frac{C(n)}{n} = AC(n)$, so that $C(n) = 100 + 50n + 5n^3$. So, the fixed cost is 100, and the marginal cost is $C'(n) = 50 + 15n^2$. □

Exercise 8.3. Suppose $R(t)$ is a (continuous) stream of income at any time $t \geq 0$ going into a bank account. If the bank account earns interest rate r, then after T years, the account has
$$\int_0^T R(t)e^{rt}dt$$
dollars in it.

- Suppose $r = .03$ and $R(t) = 5000$. How much money is in the account after $T = 10$ years?

- Suppose $r = .03$ and R is defined by

$$R(t) = \begin{cases} 4000 & , 0 \leq t \leq 3 \\ 6000 & , 3 < t \leq 6 \\ 6000e^{.03(t-6)} & , 6 < t \leq 10. \end{cases}$$

  How much money is in the account after $T = 10$ years?

The first case corresponds to putting money in the account at a roughly constant rate, whereas the second case corresponds to increasing your deposits into the account over time.

In each case, what is the present value of the money that is in the account ten years from now?

Solution. In the first case, the money in the account after ten years is

$$\int_0^{10} 5000e^{.03t}dt = \frac{1}{.03}5000e^{.03t}|_{t=0}^{t=10} = \frac{5000}{.03}(e^{.3} - 1).$$

In the second case the amount of money is

$$\begin{aligned} &\int_0^3 4000e^{.03t}dt + \int_3^6 6000e^{.03t}dt + \int_6^{10} 6000e^{.03(t-6)}e^{.03t}dt \\ &= \frac{1}{.03}4000e^{.03t}|_{t=0}^{t=3} + \frac{1}{.03}6000e^{.03t}|_{t=3}^{t=6} + \int_6^{10} 6000e^{.06t-.09}dt \\ &= \frac{1}{.03}4000(e^{.09} - 1) + \frac{1}{.03}6000(e^{.18} - e^{.09}) + \frac{1}{.06}6000e^{.06t-.18}|_{t=6}^{t=10} \\ &= \frac{1}{.03}4000(e^{.09} - 1) + \frac{1}{.03}6000(e^{.18} - e^{.09}) + \frac{1}{.06}6000(e^{.42} - e^{.18}) \end{aligned}$$

In either case, to compute the present value of the money, we multiply the above numbers by $e^{-10(.03)} = e^{-.3}$. □

Exercise 8.4. A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate $f = f(t) \geq 0$, where t is the time measured in months since its last overhaul. Because a fixed cost $A > 0$ is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.

(a) Explain why $\int_0^t f(s)ds$ represents the loss in value of the machine over the period of time t since the last overhaul.
(b) Let $C = C(t)$ be given by

$$C(t) = \frac{1}{t}\left[A + \int_0^t f(s)ds\right].$$

What does C represent and why would the company want to minimize C?
(c) Assume that $\lim_{s\to\infty} f(s) = \infty$. Show that C has a minimum value at one of the numbers $t = T$ where $C(T) = f(T)$.

Solution. $C(t)$ is the average cost of running the machine from time 0 to time t (right before the next overhaul occurs). For this reason, the company should minimize C.

From the product rule and the second part of the fundamental theorem of calculus,

$$C'(t) = -t^{-2}\left[A + \int_0^t f(s)ds\right] + t^{-1}\left[f(t)\right].$$

So, if $C'(t) = 0$, then $f(t) = \frac{1}{t}\left[A + \int_0^t f(s)ds\right] = C(t)$. Note that $\lim_{t\to 0^+} C(t) \geq \lim_{t\to 0^+} \frac{A}{t} = \infty$. So, the minimum value of C on $[0,\infty)$ occurs at a critical point as long as $\lim_{t\to\infty} C(t) = \infty$. And it is true that $\lim_{t\to\infty} C(t) = \infty$, since for any $0 < t$, we have (using our properties of integrals),

$$C(t) \geq \frac{1}{t}\int_0^t f(s)ds \geq \frac{1}{t}\int_{t/2}^t f(s)ds \geq \frac{1}{t}(t - t/2)\min_{s\in[t/2,t]} f(s) = \frac{1}{2}\min_{s\in[t/2,t]} f(s).$$

And the last quantity goes to infinity as $t \to \infty$. □

Exercise 8.5. A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate $f = f(t)$ and will accumulate maintenance costs at the rate $g = g(t)$, where t is the time measure in months. The company wants to determine the optimal time to replace the system.

(a) Let
$$C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]ds.$$
Show that the critical points of C occur at the numbers t where $C(t) = f(t) + g(t)$.

(b) Suppose
$$f(t) = \begin{cases} \frac{V}{15} - \frac{V}{450}t & \text{, if } 0 < t \leq 30 \\ 0 & \text{, if } t > 30 \end{cases},$$
and suppose $g(t) = \frac{Vt^2}{12900}$ for $t > 0$. Determine the length of time T for the total depreciation $D(t) = \int_0^t f(s)ds$ to equal the initial value V.

(c) Determine the absolute minimum of C on $(0, T]$.

(d) Sketch the graphs of C and $f+g$ in the same coordinate system, and verify the result of part (a) in this case.

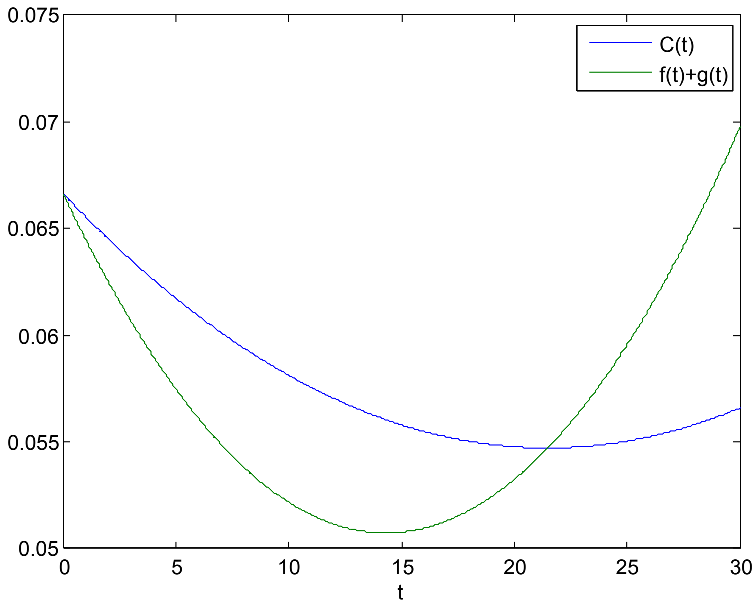
Solution. From the Fundamental Theorem of Calculus and the product rule, $C'(t) = t^{-1}(f(t) + g(t)) - t^{-2}\int_0^t f(s) + g(s)ds = t^{-1}(f(t) + g(t)) - t^{-1}C(t)$. So, if $C'(t) = 0$ and $t > 0$, we have $f(t) + g(t) = C(t)$.

With f as described above, we have for $0 < t < 30$, $\int_0^t f(s)ds = \int_0^t (V/15 - Vs/450)ds = [Vs/15 - Vs^2/900]_{s=0}^{s=t} = V[t/15 - t^2/900]$. So, $D(t) = V$ when $t/15 - t^2/900 = 1$, i.e. when $60t - t^2 - 900 = 0$, i.e. when $(30 - t)^2 = 0$. So, $D(T) = V$ when $T = 30$.

When $0 < t < 30 = T$, we have $\int_0^t (f(s) + g(s))ds = V[t/15 - t^2/900] + \int_0^t Vs^2/12900\, ds = V[t/15 - t^2/900 + t^3/38700]$, so that $C(t) = V[1/15 - t/900 + t^2/38700]$. We need to solve for $C(t) = f(t) + g(t) = V[1/15 - t/450 + t^2/12900]$. That is, we have $-t/900 + 2t^2/38700 = 0$.

So, either $t = 0$ or $t = 38700/1800 = 387/18 = 43/2$. So, the critical points of C occur at $t = 0$ and $t = 43/2$. Note that $C'(t) < 0$ when $0 < t < 43/2$ and $C'(t) > 0$ when $t > 43/2$. So, $t = 43/2$ is the absolute minimum of $C(t)$

Below is a plot of $C(t)$ and $f(t) + g(t)$ for $0 \le t \le 30$ and $V = 1$.



□

Exercise 8.6. Using integration by substitution, compute the following integrals

- $\int te^{t^2} dt$.
- $\int_0^1 x^3(x^4 + 1)^6 dx$.

Solution. Substituting $u = t^2$ so that $du = 2tdt$, we have

$$\int te^{t^2} dt = \int e^u (1/2) du = (1/2)e^u + C = (1/2)e^{t^2} + C.$$

Substituting $u = x^4 + 1$ so that $du = 4x^3 dx$, we have

$$\int_0^1 x^3(x^4 + 1)^6 dx = \int_{u(0)}^{u(1)} (1/4)u^6 du = \int_1^2 (1/4)u^6 du = (1/28)[u^7]_{u=1}^{u=2} = \frac{2^7 - 1}{28}.$$

□

Exercise 8.7. Using integration by parts, compute the following integrals

- $\int xe^x dx$.
- $\int_2^4 x(\ln x)^2 dx$.

Solution. We use $\int udv = uv - \int vdu$ where $v = e^x$, $u = x$ so that $dv = e^x dx$ and $du = dx$, so that

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C.$$

We use $u = (\ln x)^2$ and $v = x^2/2$, so that $du = x^{-1} 2 \ln x dx$ and $dv = xdx$, so that

$$\int_2^4 x(\ln(x))^2 dx = \int_2^4 (\ln x)^2 (d/dx)(x^2/2) dx = [(\ln x)^2 (x^2/2)]_2^4 - \int_2^4 x \ln(x) dx$$

We now integrate by parts again with $u = \ln x$ and $v = x^2/2$, so that $du = x^{-1}dx$ and $dv = xdx$, so that

$$\begin{aligned}\int_2^4 x(\ln(x))^2 dx &= 8(\ln 4)^2 - 2(\ln(2))^2 - \int_2^4 \ln(x)(d/dx)(x^2/2)dx \\ &= 8(\ln 4)^2 - 2(\ln(2))^2 - [(x^2/2)\ln(x)]_2^4 + \int_2^4 (x/2)dx \\ &= 8(\ln 4)^2 - 2(\ln(2))^2 - 8\ln(4) + 2\ln(2) + (1/4)(16-4).\end{aligned}$$

$\square$

9. Quiz 9

Exercise 9.1. Compute the following integral

$$\int \frac{xdx}{(x^2-1)^{3/2}}.$$

Solution. Substituting $u = x^2 - 1$ so that $du = 2xdx$, we have $\int \frac{xdx}{(x^2-1)^{3/2}} = (1/2)\int u^{-3/2} du = -u^{-1/2} = -(x^2-1)^{-1/2}$. $\square$

Exercise 9.2. Compute

$$\int_{-1}^1 \sqrt{|x|}\, dx.$$

Solution. $\int_{-1}^1 \sqrt{|x|}\, dx = \int_0^1 x^{1/2} dx + \int_{-1}^0 (-x)^{1/2} dx = \int_0^1 x^{1/2} dx - \int_1^0 x^{1/2} dx = 2\int_0^1 x^{1/2} dx = 2(2/3) = 4/3$, using the substitution $u = -x$. $\square$

Exercise 9.3. Find the length of the following vector: $(1,2,3) + 2 \cdot (2,3,4)$.

Solution. The vector has length

$$||(1,2,3) + 2\cdot(2,3,4)|| = ||(1,2,3) + (4,6,8)|| = ||(5,8,11)|| = \sqrt{25+64+121} = \sqrt{210}.$$

$\square$

Exercise 9.4. Find the unit vector which points in the same direction as $(1,0,3)$.

Solution. The unit vector is $\dfrac{(1,0,3)}{||(1,0,3)||} = \dfrac{(1,0,3)}{\sqrt{1+9}} = \dfrac{1}{\sqrt{10}}(1,0,3)$. □

Exercise 9.5. A person is standing on the earth. She walks 1 mile south. She then turns right and walks 1 mile west. She then turns right and walks 1 mile north. She has now returned to her starting location. What is her starting location?

Solution. She started at the north pole. At the north pole, any direction that you face is south. So, she walks 1 mile south. Then, she walks around a line of latitude. (Every point on this line of latitude is 1 mile from the north pole.) Finally, she walks 1 mile north, returning to the north pole.

Alternatively, consider a line of latitude near the south pole where the line of latitude is 1 mile in circumference (or 1/2 mile in circumference, or $1/n$ of a mile in circumference, where n is a positive integer.) Then it is possible that she started 1 mile north of this line of latitude. In this case, when she walks west, she will return to her initial intersection with the line of latitude. □

Exercise 9.6. In this problem, we can treat velocities as vectors. For example, if an airplane is moving forward at 500 miles per hour (mph), then we can represent the velocity as a vector v of length 500 pointing out of the nose of the plane. Then the speed of the airplane is the length of its velocity vector. In this way, two velocities add. For example, if w is a vector, and if the velocity of the wind itself is w, then relative to the ground, the airplane will move with velocity $v+w$.

Suppose the airplane is traveling at 500 mph and a gust of wind blows from the right of the plane to its left with a speed of 100 mph. What is the new speed of the airplane now?

Solution. The airplane has velocity vector $(500,0)$, and the wind has velocity vector $(0,100)$, so the plane now has velocity vector $(500,100)$. So, its new speed is $||(500,100)|| = \sqrt{500^2+100^2} = 100\sqrt{5^2+1} = 100\sqrt{26} \approx 509.9$ mph □

Exercise 9.7.

- Sketch the domain in $\mathbf{R}^2$ which is the set of all (x,y) such that $1 \leq x^2+y^2 \leq 4$.
- Sketch the domain in $\mathbf{R}^3$ which is the set of all (x,y,z) such that $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$.
- Sketch the domain in $\mathbf{R}^3$ which is the set of all (x,y,z) such that $x^2+y^2 \leq 4$, $z \geq 0$ and $y+z \leq 5$.

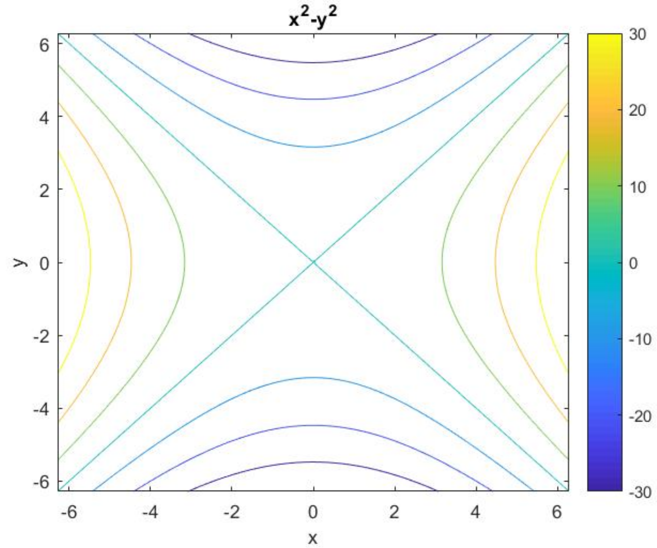
Solution.

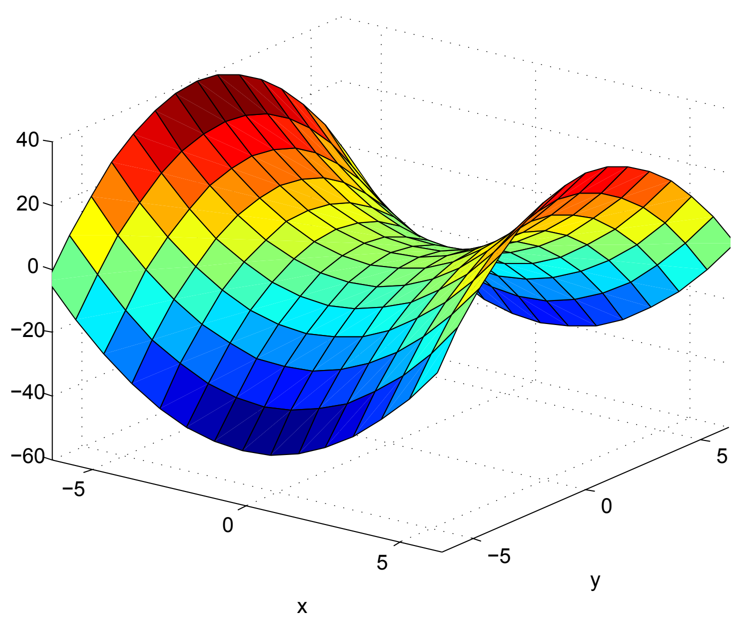
- The first domain is an annulus. It is the set of all points lying between the circle of radius 1 and the circle of radius 2, both centered at the origin.
- The second domain is a solid box of volume 1. The boundary of the domain consists of six squares each of area 1.
- This domain is a solid cylinder, with the top chopped off at an angle. It is something like a tree stump, where the top of the stump has been cut at an angle relative to the ground.

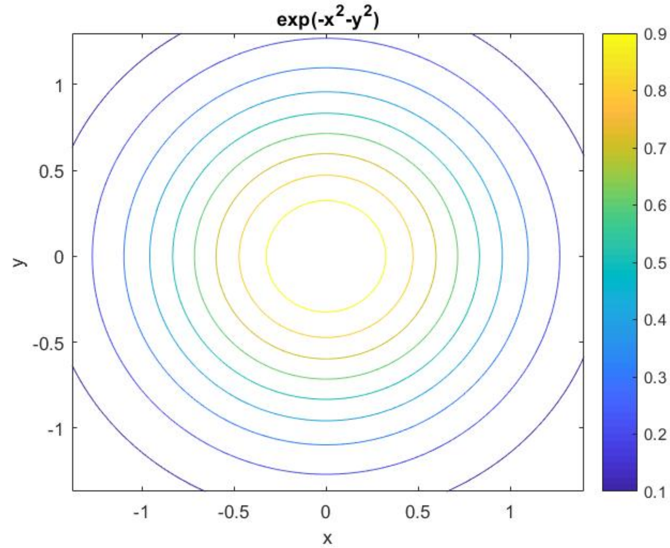
□

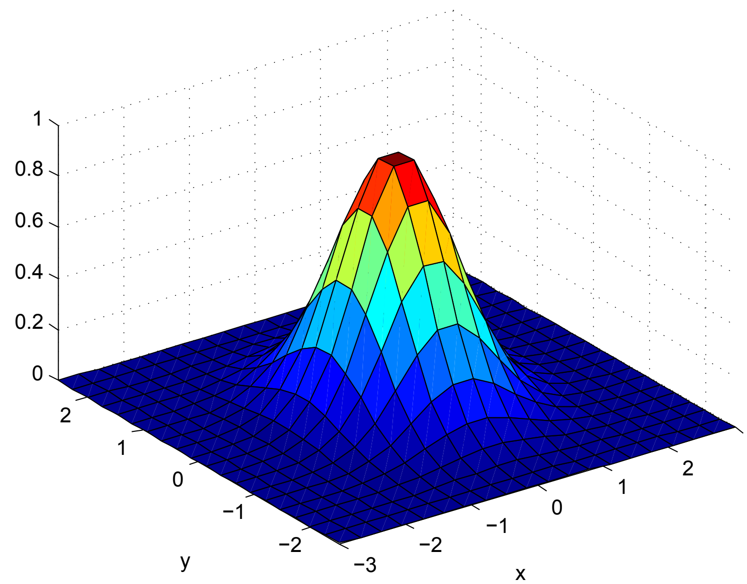
Exercise 9.8.

- Sketch the function $z = f(x, y) = x^2 - y^2$ using a contour plot.
- Sketch the function $z = f(x, y) = e^{-(x^2+y^2)}$ using a contour plot.
- Sketch the function $z = f(x, y) = 1/(xy)$ using contour or surface plots.









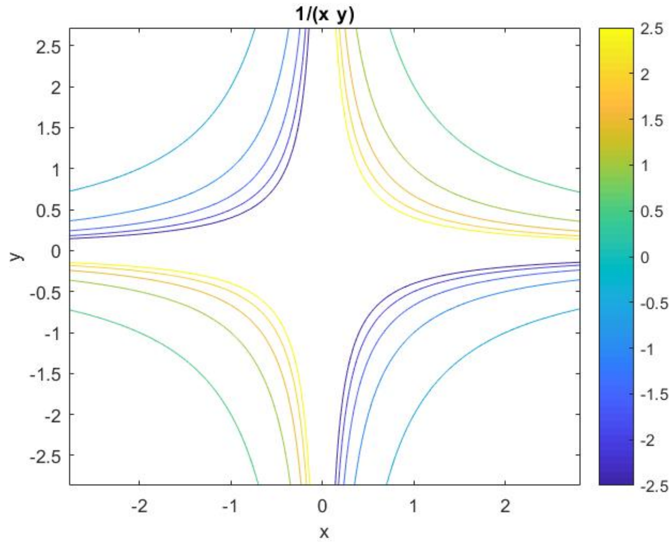
10. Quiz 10

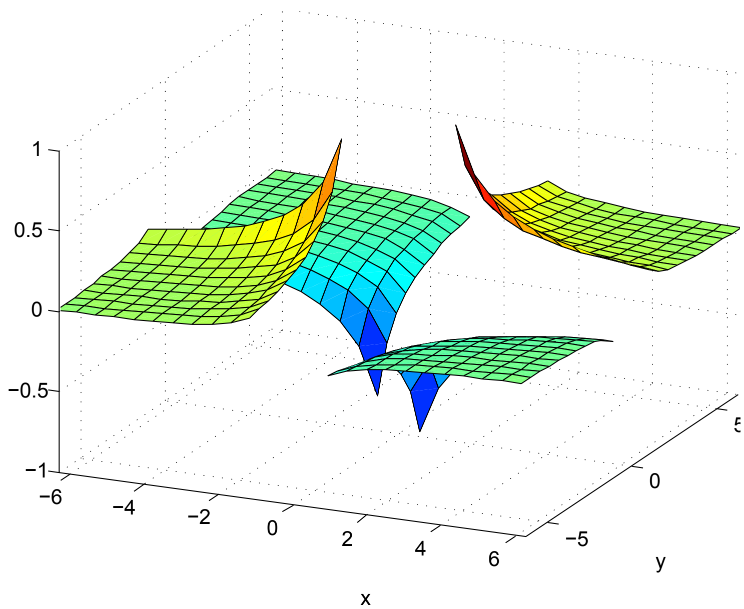
Exercise 10.1. Let $f(x, y) = x^2 + y^2$. Compute the partial derivatives: f_{xx}, f_{xy}, f_{yx}, f_{yy}.

Solution. We have $f_x = 2x$ and $f_y = 2y$, so $f_{xx} = 2$, $f_{xy} = 0$, $f_{yx} = 0$ and $f_{yy} = 2$. □

Exercise 10.2. Let $f(u, v, w, x, y, z) = u^2/v + vxyz + e^{xwv}$. Compute the partial derivatives: f_{uv}, f_{wz}, f_{xyz}.

Solution. We have $f_u = 2u/v$, $f_w = xve^{xwv}$ and $f_y = vxz$, so $f_{uv} = -2uv^{-2}$, $f_{wz} = \frac{\partial}{\partial z} f_w = 0$, and $f_{xyz} = v$. □





Exercise 10.3. Consider the following function $f\colon \mathbf{R}^2 \to \mathbf{R}$.

$$f(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}, \qquad t > 0.$$

Show that f satisfies the **heat equation** (for one spatial dimension x):

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

The function f represents a single point of heat emanating through an infinite rod (the x-axis) as time passes (as t increases, $t \geq 0$). The heat equation roughly says that the rate of change of heat f at the point x and at time t is equal to the average difference between the current heat at x, and the neighbors of x. The quantity $\partial f/\partial t$ is the rate of change of heat, while the second derivative on the right is perhaps better understood using the second-difference quotient:

$$\partial^2 f/\partial x^2 = \lim_{h\to 0} \frac{f(x-h,t) - 2f(x,t) + f(x+h,t)}{h^2}.$$

Solution. We have $f_x = t^{-1/2}(-2x)(4t)^{-1}e^{-x^2/(4t)} = -(1/2)t^{-3/2}xe^{-x^2/(4t)}$, so

$$\begin{aligned} f_{xx} &= -(1/2)t^{-3/2}\left(x(-2x/(4t))e^{-x^2/(4t)} + e^{-x^2/(4t)}\right) \\ &= (1/4)t^{-5/2}x^2e^{-x^2/(4t)} - (1/2)t^{-3/2}e^{-x^2/(4t)}. \end{aligned}$$

Also,

$$\begin{aligned} f_t &= t^{-1/2}(-x^2/4)(-t^{-2})e^{-x^2/(4t)} + e^{-x^2/(4t)}(-1/2)t^{-3/2} \\ &= t^{-5/2}x^2(1/4)e^{-x^2/(4t)} - (1/2)t^{-3/2}e^{-x^2/(4t)}. \end{aligned}$$

Therefore, $f_t = f_{xx}$. □

Exercise 10.4. Let $f(x,y) = x^2+y^3$. Compute the gradient $\nabla f(x,y)$. Find the linearization of f at the point $(a,b) = (1,2)$. Using this linearization and the approximation $f(x,y) \approx L(x,y)$, approximate the quantity $f(1.1, 1.9)$.

Solution. We have $L(x,y,z) = f(a,b,c) + ((x,y,z)-(a,b,c))\cdot\nabla f(a,b,c) = 12 + ((x,y,z) - (1,2,3))\cdot(2,12,1)$. So, $f(1.1,1.9,3.2) \approx L(1.1,1.9,3.2) = 12 + (.1,-.1,.2)\cdot(2,12,1) = 12 + 2/10 - 12/10 + 2/10 = 12 - 8/10 = 112/10$. □

Exercise 10.5. Let $f(x,y) = x^2y^3$. Compute the gradient $\nabla f(x,y)$. Then, find the tangent plane to the surface $z = f(x,y)$ at the point $(a,b) = (1,2)$.

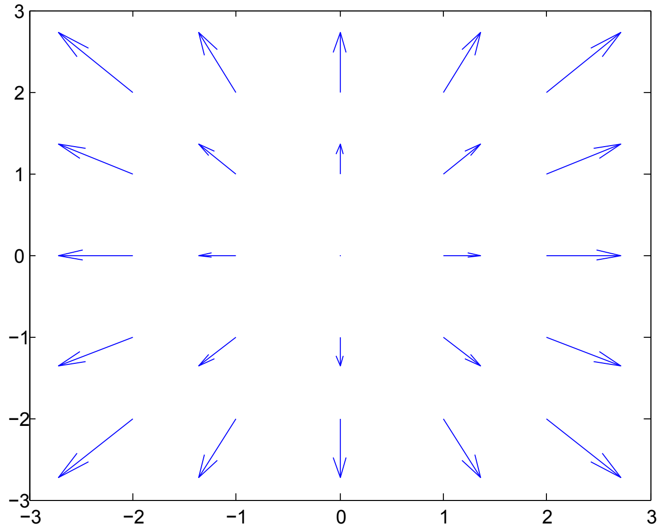
Solution. We have $\nabla f(x,y) = (2xy^3, 3x^2y^2)$, so that $L(x,y) = f(a,b) + ((x,y)-(a,b))\cdot\nabla f(a,b) = 8 + ((x,y)-(1,2))\cdot(16,12)$. The tangent plane at (a,b) is given by the equation $z = L(x,y) = 8 + ((x,y)-(1,2))\cdot(16,12)$. □

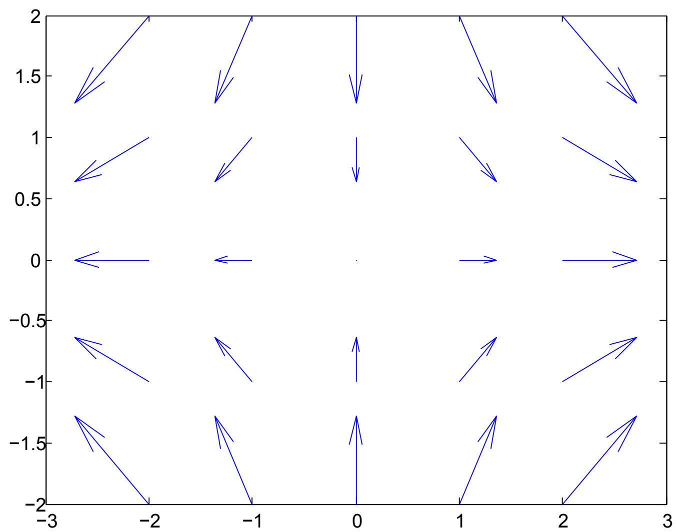
Exercise 10.6. Let $f(x,y) = x^2+y^2$ and let $g(x,y) = x^2-y^2$. For any point (x,y) in the plane, we can plot the vector $\nabla f(x,y)$ in the plane, so that $\nabla f(x,y)$ has basepoint (x,y). Plotting the vector $\nabla f(x,y)$ in this way for many values of (x,y) allows us to visualize the gradient $\nabla f(x,y)$. For any x in the set $-2,-1,0,1,2$, and for any y in the set $-2,-1,0,1,2$, plot the gradient vector $\nabla f(x,y)$. Then, in a separate drawing, plot the gradient vector $\nabla g(x,y)$.

We plot the gradients $\nabla f(x,y) = (2x,2y)$ and $\nabla g(x,y) = (2x,-2y)$. (The lengths of the vectors in these plots are not to scale.)

Exercise 10.7. It is the zombie apocalypse. It is safer at the moment to run to higher ground. The height of the land nearby is proportional to the function $f(x,y) = e^{-(x^2+y^2)/2} + xy^3$. You are located at the point $(x,y) = (1,-1)$. In which direction should you run if you want to immediately:

- Move to higher ground?
- Stay at the same elevation?





- Move to lower ground?

Solution. We have $\nabla f(x,y) = (-xe^{-(x^2+y^2)/2} + y^3, -ye^{-(x^2+y^2)/2} + 3xy^2)$. So, let $v = \nabla f(1,1) = (-e^{-1} - 1, e^{-1} + 3)$. If we want to move to higher ground, we should run in the direction v; if we want to move to lower ground, we should run in the direction $-v$; if we want to stay at the same elevation, we should run in any direction perpendicular to v. $\square$

Exercise 10.8. Let $f(x,y) = x^2 + xy + y$. Identify the critical points of f, and identify these points as local maxima, local minima or saddle points.

Solution. We have $\nabla f(x,y) = (2x + y, x + 1)$. Suppose $\nabla f(x,y) = (0,0)$. Then $2x + y = 0$ and $x + 1 = 0$. So, $x = -1$ and $y = 2$. That is, the only critical point of f occurs at $(-1,2)$. At this point, we have $f_{xx}(-1,2)f_{yy}(-1,2) - (f_{xy}(-1,2))^2 = 2(0) - (1)^2 = -1$. So, the point $(-1,2)$ is a saddle point. $\square$

Exercise 10.9. Find the maximum and minimum values of the function $f(x, y) = e^{-x^2-y^2}$ in the plane, if they exist. If they do not exist, briefly explain why.

Solution. We have $\nabla f(x, y) = f(x, y)(-2x, -2y)$. So, $\nabla f(x, y) = (0, 0)$ only when $(x, y) = (0, 0)$. We have $f(0, 0) = 1$. No other critical points of f exist. Note that $f(x, y) = e^{-x^2-y^2} \leq e^0 = 1$, since the argument in the exponential is always nonpositive. In summary, we always have $f(0, 0) \geq f(x, y) > 0$ for any (x, y), and $\lim_{x\to\infty} f(x, 0) = 0$. So, no minimum value of f exists in the plane, but the maximum value is 1. □

USC Department of Mathematics, Los Angeles, CA 90089-2532

E-mail address: stevenmheilman@gmail.com