Please provide complete and well-written solutions to the following exercises.

No due date, but the quiz in Week 6 in the discussion section (on September 27) will be based upon this homework.

Q5: Quiz 5 Problems

Exercise 1. In chemistry, a "second order" reaction satisfies $dy/dt = k(y(t))^2$ where $y(t)$ is the concentration of some chemical at time t. If $y_0 = y(0) \neq 0$, verify that we must have

$$y(t) = \frac{1}{y_0^{-1} - kt}.$$

Exercise 2 (**Newton's Law of Cooling**)**.** Suppose $y(t)$ is the temperature of an object at time t. If an object is of a different temperature than its surroundings, then the rate of change of the object's temperature is proportional to the difference of the temperature of the object and the temperature of the surroundings. That is, if Y denotes the temperature of the surroundings, and if $y(0) = y_0 \neq Y$, then there exists a constant $k > 0$ such that

$$y'(t) = -k(y(t) - Y).$$

Note that if $y(0) < Y$, then $y'(0) > 0$, so that the temperature of y is increasing to the environment's temperature. And if $y(0) > Y$, then $y'(0) < 0$, so that y is decreasing to the environment's temperature.

Let $f(t) = y(t) - Y$. Verify that $f'(t) = -kf(t)$. Conclude that $f(t) = y(t) - Y = (y_0 - Y)e^{-kt}$. That is, we have Newton's Law of cooling:

$$y(t) = Y + (y_0 - Y)e^{-kt}.$$

Exercise 3. The exponential growth model for bacteria is a bit unrealistic, since after a while, the bacteria are limited by their environment and food supply. We therefore consider the **logistic growth** model. Suppose $y(t)$ is the amount of bacteria in a petri dish at time t and $k > 0$ is a constant. Let C be the maximum possible population of the bacteria. We model the growth of the bacteria by the formula

$$y'(t) = ky(t)(C - y(t)), \qquad y(0) = y_0$$

So, when y is small, $y'(t)$ is proportional to y. However, when y becomes close to C, y' becomes very small. That is, the rate of growth of bacteria is constrained by the environment.

- Verify that the following function satisfies the above differential equation.

$$y(t) = \frac{C}{1 + (Cy_0^{-1} - 1)e^{-ktC}}.$$

- Plot the function $y(t)$. (What are the limits of y as t goes to $+\infty$ and $-\infty$?)
- Find out where $y'(t)$ is the largest. (Hint: find the maximum of the function of y: $ky(C - y)$.) (This point t is called the point of **diminishing returns** in economics.)

The latter observation explains the "J-curve" scare for human population growth in the 1980s. At this point in time, many people were afraid that the human population would grow too large for the earth to support us. However, it seems that we were simply observing the maximum possible growth rate of the human population at this time (if we believe that logistic growth models the human population reasonably well).

Exercise 4. Let $f(x) = (1+x)^{15}$. Near $x = 0$, show that the linear approximation of f is given by $1 + 15x$, meaning $(1+x)^{15} \approx (1 + 15x)$ when x is near zero.

Exercise 5. Let $a, b > 0$. Find the maximum value of $f(x) = x^a(1-x)^b$ on the interval $[0, 1]$.

Exercise 6. Let $a, b > 0$. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$. You may assume that the rectangle is aligned with the axes, and that its vertices touch the ellipse.

Exercise 7. Find the minimum and maximum values of $f(x) = 2\sqrt{x^2+1} - x$ on the interval $[0, 2]$

Exercise 8. It is the zombie apocalypse. You are in the forest, five miles from a straight road. If you traveled in a straight line directly towards the road from your current position (which is not a good idea), you would then have to walk 10 miles along the road to get to the safe house. You can travel at two miles per hour in the forest, and you can travel at four miles per hour on the road. What is the shortest amount of time that it will take to get to the safe house at the end of the road?